

The strange properties of the infinite power tower

An “investigative math” approach for young students

Luca Moroni*

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Nevertheless, the fact is that there is nothing as dreamy and poetic, nothing as radical, subversive, and psychedelic, as mathematics.

Paul Lockhart – “*A Mathematician’s Lament*”

Abstract

In this article we investigate some “unexpected” properties of the “*Infinite Power Tower*¹” function (or “*Tetration with infinite height*”):

$$y = f(x) = x^{x^{x^{x^{\dots}}}}$$

where the “tower” of exponentiations has an infinite height.

Apart from following an initial personal curiosity, the material collected here is also intended as a potential guide for teachers of high-school/undergraduate students interested in planning an activity of “*investigative mathematics in the classroom*”, where the knowledge is gained through the active, creative and cooperative use of diversified mathematical tools (and some ingenuity).

The activity should possibly be carried on with a laboratorial style, with no preclusions on the paths chosen and undertaken by the students and with little or no information imparted from the teacher’s desk.

The teacher should then act just as a guide and a facilitator.

The infinite power tower proves to be particularly well suited to this kind of learning activity, as the student will have to face a challenging function defined through a rather uncommon infinite recursive process. They’ll then have to find the right strategies to get around the trickiness of this function and achieve some concrete results, without the help of *pre-defined* procedures.

The mathematical requisites to follow this path are: functions, properties of exponentials and logarithms, sequences, limits and derivatives. The topics presented should then be accessible to undergraduate or “advanced high school” students.

keywords — *infinite power tower, tetration, fixed-point, recursion, recursive sequence, cobweb, Euler, Lambert, Lagrange*

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*Liceo Scientifico “Donatelli-Pascal” - Milan - Italy

¹We adopt here the quite popular term *power tower* even if it is not entirely correct. The expression should instead be described in terms of *exponentiations* and not *powers*.

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1 Overview

After presenting the infinite power tower function, its definition and its unexpected properties (section **2** - *Introduction*), we start an investigation about its mathematical characteristics. In section **3** (*Generalization*) we introduce the function $y = x^y$ and its inverse function $x = y^{1/y}$ that prove to be useful to give some promising clues on the infinite power tower. In section **4** (*The problem of convergence*) we introduce the problem of the convergence of the recursive sequence leading to the infinite power tower. This problem is furtherly investigated in the sections **5** (*Fixed points and convergence criteria (in general)*), **6** (*Fixed points and convergence of the power tower (the algebraic route)*) and **7** (*Fixed points and convergence of the power tower (the graphical route)*), where the investigation is brought forward with both algebraic and graphical methods. Section **8** (*Outside the convergence interval*) explores what happens outside the convergence interval and the emergence of a periodic cycle for the values given by the power tower function. Lastly, in section **9** (*Some history about the power tower*) we briefly discuss some very interesting historical aspects on the origin of the interest about the infinite power tower (where the main characters are Lambert, Euler and Lagrange).

2 Introduction

Let's define the “*Infinite Power Tower*” function (or “*Tetration with infinite heights*”) as:²

$$y = f(x) = x^{x^{x^{\dots}}}$$

where the tower of exponentiations has an infinite height.

People aware of the *explosive* nature of exponential functions will guess that, if $x > 1$, the $f(x)$ previously defined will soon blow up to infinity as the height of the tower is increased. But, contrary to this initial guess, some trial with a pocket calculator suggests that there might be a stable behavior for some set of values, even with $x > 1$.

In fact, some numerical experiments show that if we set $x = \sqrt{2}$, then

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\dots}}}} \rightarrow 2$$

The reason for that can be gained by the following reasoning. Since the sequence of exponentials is infinite, adding (or removing) one element to an infinite sequence shouldn't change its overall effect (like adding or subtracting a finite number to infinite). We can then follow the passages outlined below:

$$y = \sqrt{2}^{\left\{ \sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\dots}}} \right\}=y} \rightarrow y = (\sqrt{2})^y \rightarrow y^{1/y} = \sqrt{2} = 2^{1/2} \rightarrow y = 2$$

We could be tempted to extend and generalize the procedure in the following way

$$y = x^{\left\{ x^{x^{x^{\dots}}} \right\}=y} \rightarrow y = x^y \rightarrow y^{1/y} = x$$

so that, setting $y = 3$ it would be $x = 3^{1/3} = \sqrt[3]{3}$ and setting $y = 4$ it would be $x = \sqrt[4]{4} = \sqrt{2}$.

But here we have a problem: if we set $x = \sqrt{2}$ what will we get for the y : 2 or 4?

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\dots}}}} = \sqrt[4]{4}^{\sqrt[4]{4}^{\sqrt[4]{4}^{\sqrt[4]{4}^{\dots}}}} \rightarrow 2? \ 4?$$

Let's check again numerically.

After having defined the following recursive function in *Mathematica* or in *Geogebra*

Mathematica: `PowerTower[a_, k_Integer] := Nest[Power[a, #] &, 1, k]`

Geogebra: `Iteration(a^x, a, n - 1)` (where a and n can be defined as sliders)

we find that a tower with height=1000, starting from $x = \sqrt{2}$, yields a result of 2 (as expected, anyway not “4”), but if the starting point is $\sqrt[3]{3}$ the result is not 3 (as previously supposed), but rather a mysterious 2.47805.

²In the rest of this article we'll assume that $x, y \in \mathbb{R}$ and $x > 0, y > 0$

```
In[90]:= PowerTower[ $\sqrt{2.}$ , 1000]
```

```
Out[90]= 2.
```

```
In[92]:= PowerTower[ $\sqrt[3]{3.}$ , 1000]
```

```
Out[92]= 2.47805
```

The same results are confirmed when the height of the exponentiations is increased to even higher values so that we can be confident that, for these values of the x , there is a definite value for the y .

Another strange thing happens when we give the x some values close to 0 and consider odd/even numbers for the height of the tower:

```
In[44]:= PowerTower[0.025, 1000]
```

```
Out[44]= 0.853652
```

```
In[45]:= PowerTower[0.025, 1001]
```

```
Out[45]= 0.0428941
```

```
In[46]:= PowerTower[0.025, 1002]
```

```
Out[46]= 0.853652
```

```
In[47]:= PowerTower[0.025, 1003]
```

```
Out[47]= 0.0428941
```

It seems that a small change in the height of the tower may produce a relevant change in the result. How can it be?

These initial experiments suggest the following practical questions to be addressed:

- Why is $f(\sqrt{2}) = 2$ (and not 4)?
- Why is not $f(\sqrt[3]{3}) = 3$?
- For which values of x do we have a definite (finite) value of y ?
- Why do we sometimes get two different values with small changes in the height of the tower?

3 Generalization

We have previously defined the infinite power tower function as:

$$y = f(x) = x^{x^{x^{\ddots}}}$$

where the tower of exponentiations has an infinite height.

Firstly, let's make clear what is the conventional meaning of applying subsequent exponentiations.

In order to do so it's convenient to start from the definition of the related functions representing towers with **finite heights**. It will be:

$$y_n = f_n(x) = x^{x^{x^{\ddots}}} \text{ }^{n \text{ times}}$$

so it is $y_1 = f_1(x) = x$, $y_2 = f_2(x) = x^x$, $y_3 = f_3(x) = x^{x^x}$ and so on.

It's important to observe that it is $x^{x^x} = x^{(x^x)}$ and not $x^{x^x} = (x^x)^x = x^{x^2}$.

This means that the tower is built from the highest exponent downwards to the lowest level.

This example shows the difference between a downwards and an upwards construction:

$$3^{(3^3)} = 3^{27} = 7\,625\,597\,484\,987 \neq (3^3)^3 = 9^3 = 729$$

Using the definition of the "finite" tower $f_n(x)$, the infinite power tower can then be re-defined as follows:

$$y = f(x) = x^{x^{x^{\ddots}}} = \lim_{n \rightarrow \infty} f_n(x)$$

Alternatively, we can build the sequence of functions $\{y_1, y_2, y_3, y_4, \dots, y_n, \dots\}$ and take advantage of the fact that this sequence can be defined recursively as:

$$\begin{cases} y_1 = x \\ y_{n+1} = x^{y_n} \end{cases} \quad f(x) = \lim_{n \rightarrow \infty} y_n$$

It's easy to check that with above definition we have

$$\begin{aligned} y_1 &= x \\ y_2 &= x^{y_1} = x^x \\ y_3 &= x^{y_2} = x^{x^x} \\ &\dots \end{aligned}$$

reproducing, when $n \rightarrow \infty$, our infinite power tower.

After having clarified the meaning of the infinite power tower function $y = f(x)$ we can say

that, if it converges to some finite value y , than it is $y = x^{\left\{x^{x^{x^{\ddots}}}\right\}=y} \rightarrow y = x^y$

The inverse function will then be

$$x = g(y) = y^{1/y} \quad (g = f^{-1})$$

Unlike $y = x^y$ (that's not the expression in explicit form of a function), this appears to be a well-defined function (although mapping $y \mapsto x$) for any value $y > 0$. So, let's study the characteristics of this function $x = g(y) = y^{1/y}$ to get some insight on the function $f(x)$ we are mostly interested in.

Note that we'll use the following useful identity in some calculation: $x = y^{1/y} = e^{\ln y^{1/y}} = e^{\ln y / y}$

Expression: $x = y^{1/y}$

Domain: $y > 0$

Limits: $\lim_{y \rightarrow 0} y^{1/y} = 0^+ \quad \lim_{y \rightarrow \infty} y^{1/y} = 1$

in fact, using the L'Hôpital's rule (H),

$$y^{1/y} = e^{\ln y / y} \text{ and } \lim_{y \rightarrow 0^+} \frac{\ln y}{y} = -\infty, \quad \lim_{y \rightarrow +\infty} \frac{\ln y}{y} \stackrel{H}{=} 0$$

First derivative: $\frac{dx}{dy} = \frac{d}{dy} (e^{\ln y / y}) = e^{\ln y / y} \left(\frac{1 - \ln y}{y^2} \right) = y^{\frac{1}{y}} \left(\frac{1 - \ln y}{y^2} \right)$

Asymptotes: the line $x = 1$ is a horizontal asymptote

Stationary points: $\frac{dx}{dy} = 0 \rightarrow 1 - \ln y = 0 \rightarrow y = e \rightarrow x = e^{1/e}$

$\frac{dx}{dy} > 0 \rightarrow 1 - \ln y > 0 \rightarrow y < e$ The point $M(y_M; x_M) = (e; e^{1/e})$ is a maximum.

Second derivative: $\frac{d^2x}{dy^2} = y^{1/y-4} (1 - 3y + (2y + \ln y - 2) \cdot \ln y)$

Inflection points: $1 - 3y + (2y + \ln y - 2) \cdot \ln y = 0 \rightarrow \begin{cases} y_{F1} \approx 0.5819, & x_{F1} \approx 0.3944 \\ y_{F2} \approx 4.3678, & x_{F2} \approx 1.4015 \end{cases}$

The plot of $g(y)$ is:

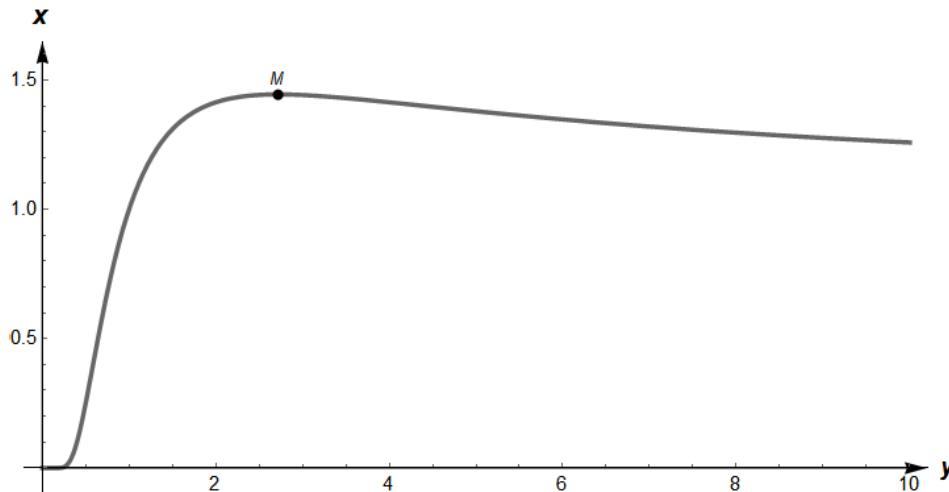


Fig. 1: Plot of $x = g(y) = y^{1/y}$

We must remember that in the plot above, differently from the usual conventions, the vertical axis represents the x and the horizontal axis is the y .

If we rotate the graph we get, now with the usual orientation of the axis, the set of points satisfying the equivalent relations:

$$x = y^{1/y} \text{ or } y = x^y \text{ or } y = e^{y \ln x}$$

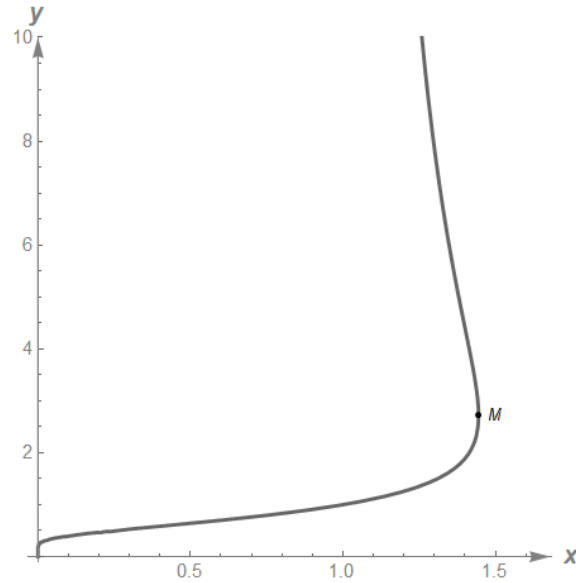


Fig. 2: Implicit plot of $y - x^y = 0$

But since $g(y)$ is not invertible as it is not bijective, the plot shown in Fig. 2 is not that of a function.

The inverse function of $g(y)$ could only be defined on a proper restriction of the domain of g , where the function is a bijection.

For example, this condition would be respected in the region defined by $0 < x \leq e^{1/e} \wedge y > 0$ and we'd get the following plot:

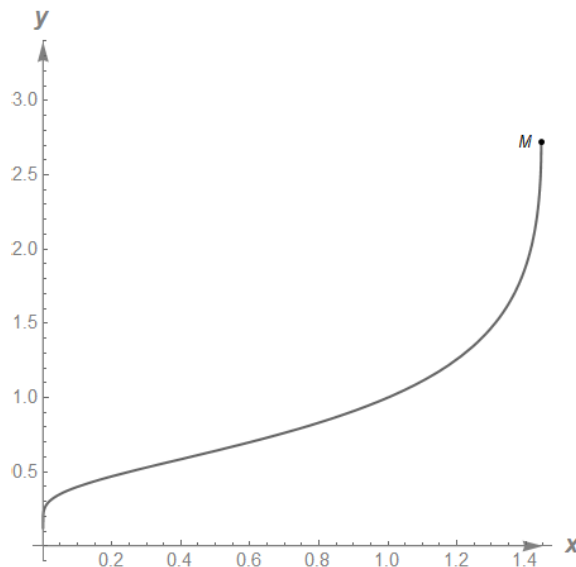


Fig. 3: Possible inverse function of $g(y)$

4 The problem of convergence

The plot of Fig. 2 represents all the points satisfying the equation $y = x^y$. Anyway, it would be problematic to say that these points are also the ones satisfying the equation of the infinite power tower

$$y = f(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$$

In fact, above equation is written in the form of a function, whilst $y = x^y$ is not the expression of a function.

Furthermore the plot tells us that for some values of the x (with $1 < x < e^{1/e}$) we would get two possible values of the y and this doesn't make much sense with how the $f(x)$ is defined.

The problem is hidden in the following passage:

If the infinite power tower converges then it is $y = x^{x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}}} = x^{\left\{ x^{x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}}} \right\} = y} \rightarrow y = x^y$

But the truth is that the infinite power tower **doesn't converge** for every values of x .

How can we tell that? And how can we find the interval of convergence?

We must recall that the function $f(x)$ can be defined by recursion as the limit of a sequence of functions with finite heights:

$$\begin{cases} y_1 = x \\ y_{n+1} = x^{y_n} \end{cases} \Rightarrow f(x) = \lim_{n \rightarrow \infty} y_n$$

So, given some value of x , we can say that the sequence $\{y_n\}$ converges if it stabilizes to some finite value as far as n is increased.

In practice, the convergence requires that $\lim_{n \rightarrow \infty} y_{n+1} = y_n$ (or $\lim_{n \rightarrow \infty} y_{n+1} - y_n = 0$).

To find the conditions assuring the convergence of a recursive sequence we abandon temporarily our power tower function and explore, in more general terms, sequences, fixed points and when a sequence is bound to converge to a fixed point.

5 Fixed points and convergence criteria (in general)

In general, given a sequence defined by its starting value y_1 and by the recursion equation $y_{n+1} = r(y_n)$, where r is a smooth function, a **fixed point** y^* is a value satisfying the equation $y^* = r(y^*)$. The name fixed point means that if $y_n = y^*$ then $y_{n+1} = r(y_n) = r(y^*) = y^*$ and the sequence will keep on re-producing the same value for all future iterations.

Once we have found the fixed point(s) of a sequence by solving the equation $y = r(y)$ we may be interested to know if a fixed point is stable (or attractive) or not.

If the fixed point is attractive then, when we start close to it, we will end up even closer. In mathematical terms we can say that, calling δ_n the distance between y^* and y_n ($\delta_n > 0$) and starting from a point $y_n = y^* \pm \delta_n$ the subsequent term will be $y_{n+1} = y^* \pm \delta_{n+1}$, and the requirement for the convergence is that $\delta_{n+1} < \delta_n \forall n$. Since it is

$\delta_{n+1} = |y_{n+1} - y^*| = |r(y_n) - r(y^*)|$ and $\delta_n = |y_n - y^*|$ we have

$$\frac{\delta_{n+1}}{\delta_n} = \frac{|r(y_n) - r(y^*)|}{|y_n - y^*|}$$

The closer we are to y^* the more above ratio will approximate the absolute value of the derivative $|r'(y^*)|$. This suggests that it's possible to use the *mean value theorem* to state that there exist a point $\xi \in (y_n, y^*)$ such that

$$|r'(\xi)| = \frac{|r(y_n) - r(y^*)|}{|y_n - y^*|}$$

In our case we can say that there exists a point $\xi \in (y_n, y^*)$ such that

$$\frac{\delta_{n+1}}{\delta_n} = |r'(\xi)|$$

Then, if there is some interval in which it is $|r'(y)| < k < 1 \quad \forall y \in (|y^* - y| < \delta_n)$ it will also be

$$\frac{\delta_{n+1}}{\delta_n} = |r'(\xi)| < k$$

and

$$|r(y_n) - r(y^*)| < k |y_n - y^*|$$

that is

$$|y_{n+1} - y^*| < k |y_n - y^*|, \quad |y_{n+2} - y^*| < k |y_{n+1} - y^*| < k^2 |y_n - y^*|$$

and so on.

We then see that if $|r'(y)| < 1$ in some neighborhood of y^* and if the starting point of the recursive sequence belongs to this same neighborhood, the distance to the fixed point reduces more and more as n is increased and we'll have $\lim_{n \rightarrow \infty} \delta_n = 0$ meaning that $\lim_{n \rightarrow \infty} y_n = y^*$.

In a more formal way, we state (without a complete and rigorous proof) the following theorem (*fixed point convergence criteria*):

If

- 1) $r(y)$ and $r'(y)$ are continuous on $[a, b]$
- 2) if $a \leq y \leq b \rightarrow a \leq r(y) \leq b$ (meaning that $r(y)$ is a *contraction mapping*)
- 3) $\lambda = \max_{a \leq y \leq b} |r'(y)| < 1$

Then

- a) There exists a unique solution $y^* \in [a, b]$ of the equation $y = r(y)$.
- b) For any initial starting value $y_0 \in [a, b]$ the sequence will converge to the unique fixed point: $\lim_{n \rightarrow \infty} y_n = y^*$

The convergence/divergence character of the fixed points can be interpreted graphically with the so called “*cobweb*” construction.

In the following Fig. 4 we have the recursion equation $y_{n+1} = r(y_n)$ plotted with y_{n+1} as a function of y_n . The fixed points are the intersections between $r(y_n)$ and the line $y_{n+1} = y_n$. Here we have two fixed points labeled P_1 and P_2 . The cobweb construction shows that P_1 is an

attractive fixed (stable) point whilst P_2 is a repulsive (unstable) fixed point. This is due to the fact that $|r'(y_{P_1})| < 1$ and $|r'(y_{P_2})| > 1$.

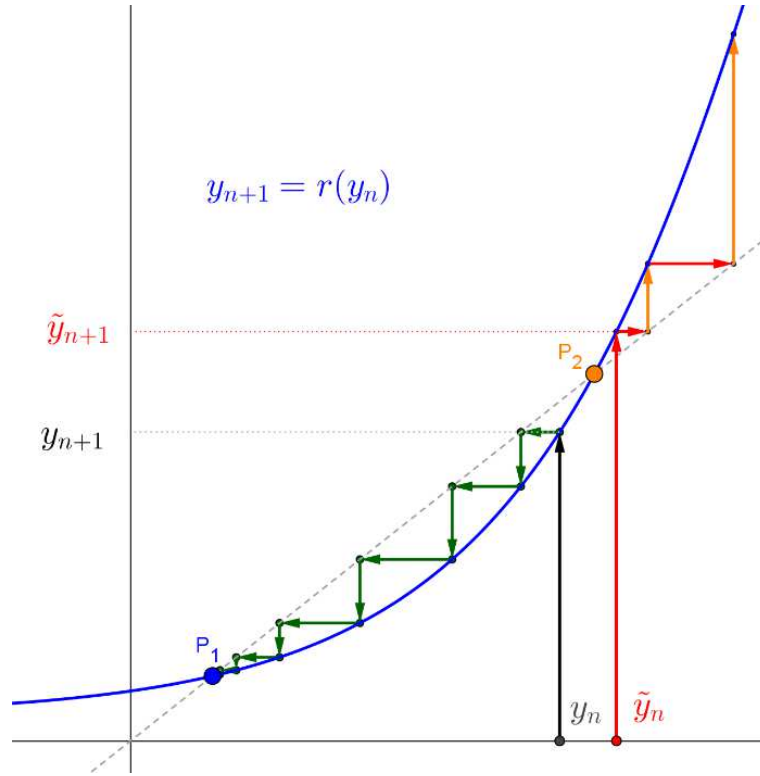


Fig. 4: Cobweb iteration of a sequence with an attractive (P_1) and a repulsive (P_2) fixed points.

6 Fixed points and convergence of the power tower (the algebraic route)

In the case presented in this article we are interested in the convergence of the sequence of functions

$$y_{n+1} = x^{y_n}.$$

Here the x variable should be considered as a parameter of the recursion equation whose variables are the terms y_n and y_{n+1} . In practice we have an infinite number of sequences, one for each value of x .

The fixed points (y^*) of these sequences are those for which it is $y_{n+1} = y_n$ that is those satisfying the equation

$$y = x^y$$

Using the *fixed point convergence criteria* we must find the interval of the values of the y (and of the x) for which the first derivative of x^y has modulus less than 1, that is

$$\left| \frac{d}{dy} (x^y) \right| < 1$$

For this purpose, it's convenient to use the equivalence

$x^y = e^{\ln x^y} = e^{y \ln x}$ and calculate the following derivative, with respect to y :

$$\frac{d}{dy} (e^{y \ln x})$$

Using the fact that for the fixed points it is $y = x^y$ we have

$$\frac{d}{dy} (e^{y \ln x}) = e^{y \ln x} \cdot \ln x = x^y \cdot \ln x = y \cdot \ln x = \ln x^y = \ln y$$

For the convergence it must then be $|\ln y| < 1$, that is

$$-1 < \ln y < 1 \rightarrow 1/e < y < e$$

The corresponding values for the x (since it is $x = y^{1/y}$) are then e^{-e} and $e^{1/e}$.

Then, if we use the fixed point convergence criteria, we can say that the convergence is assured for

$$e^{-e} < x < e^{1/e}$$

producing stable fixed points in the interval

$$1/e < y < e.$$

7 Fixed points and convergence of the power tower (the graphical route)

To gain a deepest understanding of what the previous result actually means, we now switch to another route more rich of visual elements.

In the case of the sequence $y_{n+1} = x^{y_n}$ we see that the recursion equation is a family of exponential curves (think of the “ x ” as a parameter) and the search of the possible fixed points and their stability is rather simplified, mostly because these functions are strictly monotonic (apart the banal case with $x = 1$).

In order to simplify the notation let's rename the variables as follows:

$$y_{n+1} = z, \quad y_n = y$$

Then, we want to study the family of exponential functions

$$z = x^y$$

(where the base x can be considered a parameter).

With this notation a fixed point y^* is the solution of the system $\begin{cases} z = x^y \\ z = y \end{cases}$ leading to the equation $y = x^y$.

The character of the exponential is determined by the value of its base x :

- $x > 1$: the exponential is increasing;
- $x = 1$: the exponential becomes the constant line $z = 1$ and the original infinite power tower function becomes $y = 1^{1^{\cdot^{\cdot^{\cdot}}}} = 1$;
- $x < 1$: the exponential is decreasing;

The positions of the curves defined by the recursive function $z = x^y$ with respect to the *identity line* $z = y$ allow us to determine the possible existence of fixed points.

With $x > 1$ we may have the following cases (Fig. 5):

- I) The exponential curve is always above the line: there are no fixed points.
- II) The exponential curve is tangent to the line: there is one single fixed point y_1^* (or two coincident fixed points).
- III) The exponential curve intersects the line in two points: there are two distinct fixed points y_1^* and y_2^* .

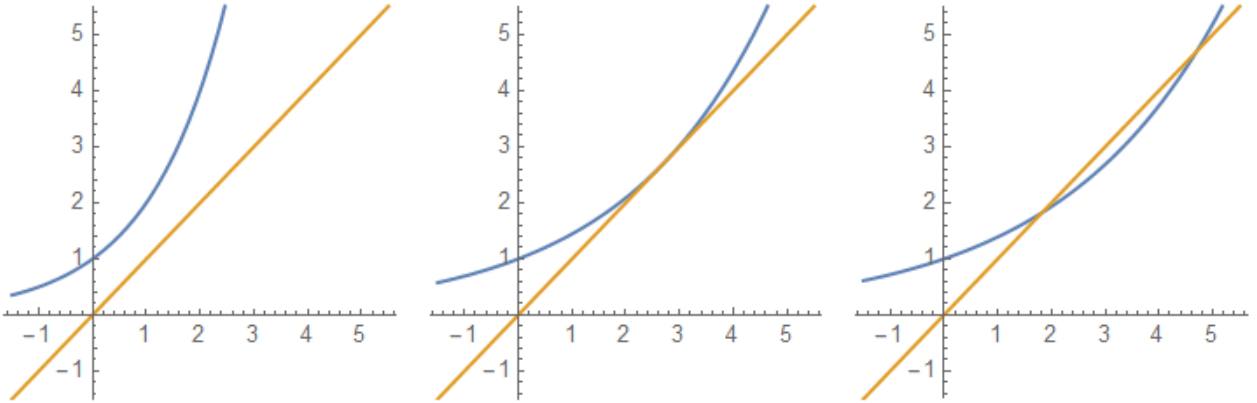


Fig. 5: : Possible relative positions of the exponential $z = x^y$ and the line $z = y$ in the case $x > 1$

With $x < 1$ (decreasing exponential) there will always be a single intersection point and a single corresponding fixed point. We'll distinguish the following cases (Fig. 6):

- IV) The first derivative in the intersection point is $-1 \leq \frac{d}{dy}(x^y) \leq 0$
- V) The first derivative in the intersection point is $\frac{d}{dy}(x^y) < -1$

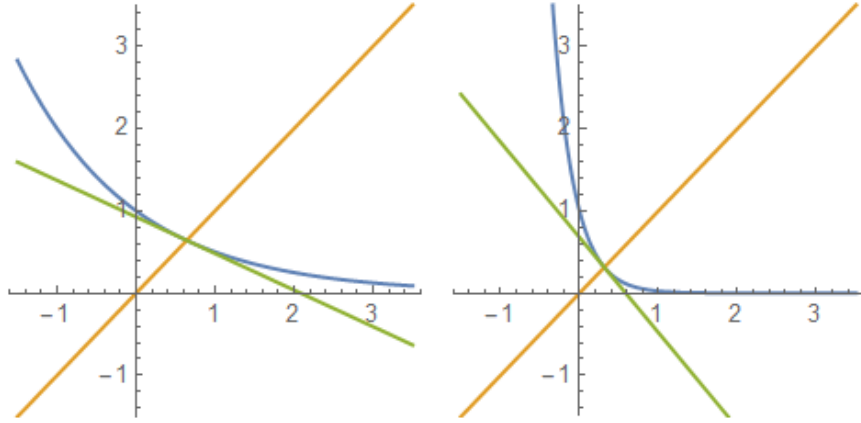


Fig. 6: The first derivative (slope of the tangent line) in the single fixed point in the case $0 < x < 1$

Now we'll examine above 5 cases, analyze the characteristics of the fixed points and find which values of the “ x ” produce them.

If $x > 1$ the discriminating case is that for which the exponential is tangent to the line (Fig. 7).

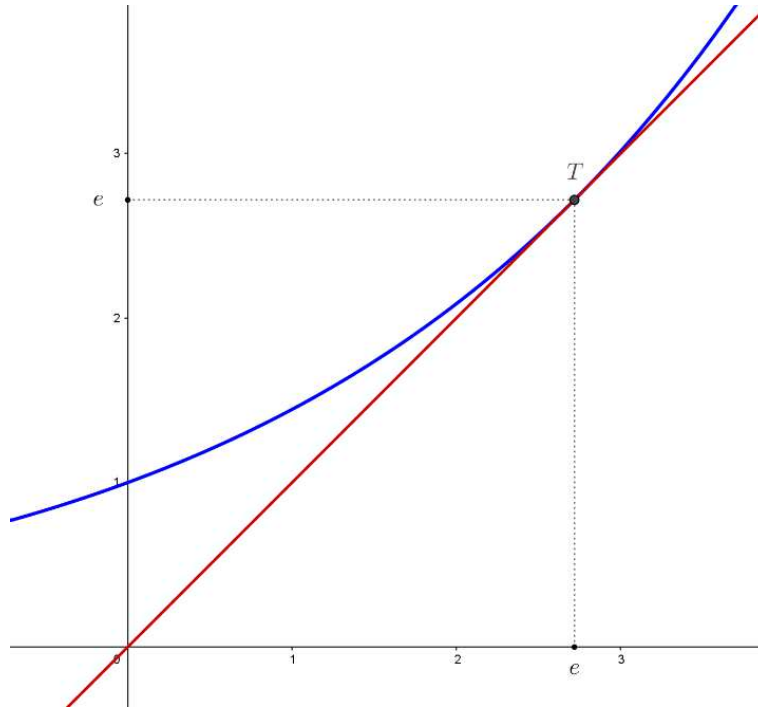


Fig. 7: : $x > 1$ with the exponential tangent to the line $z = y$

So, let's look for the tangency point T . In this point the exponential will have the same slope of the line, that is $z'(y) = \frac{d}{dy}(x^y) = 1$.

Then

$$z'(y) = \frac{d}{dy}(x^y) = \frac{d}{dy}(e^{y \ln x}) = e^{y \ln x} \cdot \ln x = x^y \cdot \ln x = 1 \rightarrow y_T = \log_x \left(\frac{1}{\ln x} \right) = -\frac{\ln \ln x}{\ln x}$$

and

$$z_T = x^{y_T} = x^{\log_x(1/\ln x)} = \frac{1}{\ln x}$$

We have found the point $T\left(-\frac{\ln \ln x}{\ln x}, \frac{1}{\ln x}\right)$ in which it is $z'(y) = 1$. But for the exponential curve to be tangent to the line $z = y$ we must impose that T belong to that line, that is

$$y_T = z_T \rightarrow -\frac{\ln \ln x}{\ln x} = \frac{1}{\ln x} \rightarrow \ln \ln x = -1 \rightarrow \ln x = e^{-1} \rightarrow x = e^{1/e} \approx 1.445.$$

With this value the exponential function becomes $z = (e^{1/e})^y$ and the point of tangency is $T(e, e)$.

Knowing how the base influences the graphic of a generic exponential curve we can also say that:

If $x > e^{1/e}$ there's no intersection (and no fixed points for the recursive sequence).

If $x = e^{1/e}$ there is a single intersection (and a single fixed point for the recursive sequence).

If $1 < x < e^{1/e}$ there are two intersections (and two fixed points for the recursive sequence).

With the cobweb diagram we can see what evolution the sequence will follow in these cases:

If $x > e^{1/e}$ (Fig. 8) there is no fixed point and, with any starting point, the sequence is bound to diverge to infinity.

If $x = e^{1/e}$ (Fig. 9) the cobweb iterations converge to $P_1 (e, e)$ if the starting value is to the left of P_1 and diverge if the starting value is to the right. We can say that P_1 is a “half-stable” saddle fixed point. Anyway, for the power tower sequence the starting value is $y_0 = x = e^{1/e}$ that is located to the left of $P_1 (e, e)$. So the sequence converge to $y^* = e$.

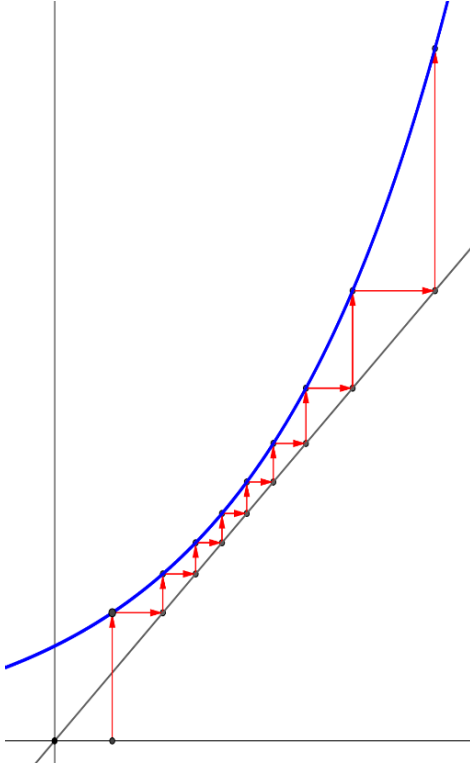


Fig. 8: Cobweb diagram in the case $x > e^{1/e}$

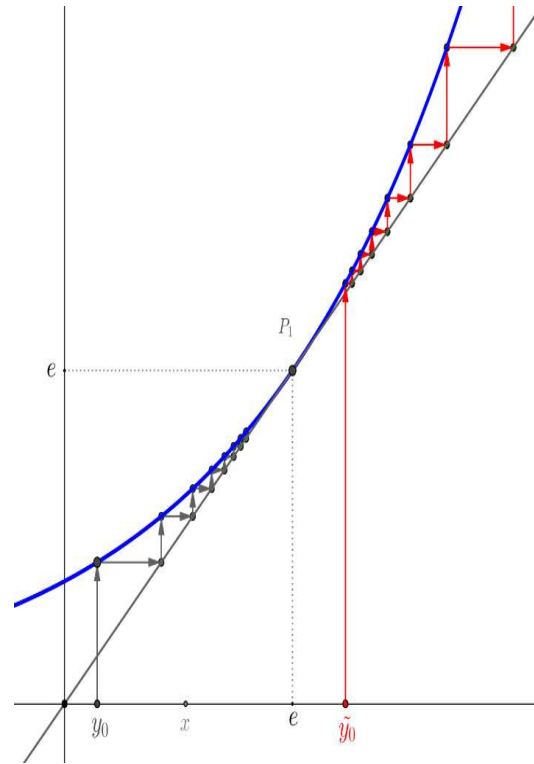


Fig. 9: Cobweb diagram in the case $x = e^{1/e}$

If $1 < x < e^{1/e}$ there are two fixed points that are the solutions of the equation $y = x^y$. Let's call them y_1^* and y_2^* (Fig. 10).

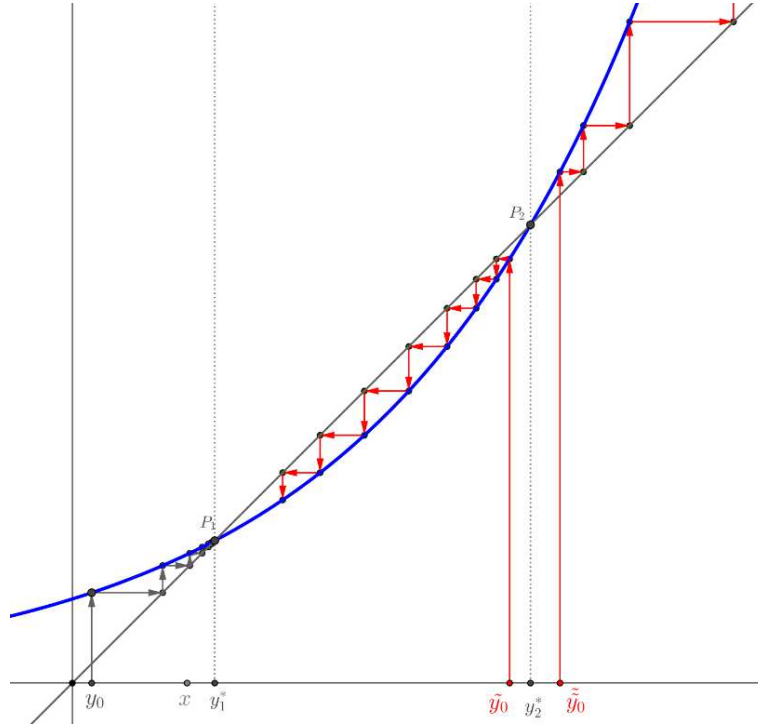


Fig. 10: Cobweb diagram in the case $1 < x < e^{1/e}$: the fixed point P_1 is attractive and the fixed point P_2 is repulsive. The actual starting point (x) is in the basin of attraction of P_1

The cobweb iterations show that y_1^* is attractive and y_2^* is repulsive. Furthermore the sequence will converge to y_1^* for any starting point $y_0 < y_2^*$ and diverge to infinity for $y_0 > y_2^*$. Anyway, for the power tower sequence the starting value is $y_0 = x$ and it's located to the left of y_1^* . In fact, for the fixed point holds the relation $y_1^* = x^{y_1^*} \rightarrow x = (y_1^*)^{1/y_1^*}$ and if we set $x < y_1^*$ it must be $(y_1^*)^{1/y_1^*} < y_1^*$. Taking the logarithms of both sides we have $\ln (y_1^*)^{1/y_1^*} < \ln y_1^*$ that is $1/y_1^* \ln (y_1^*) < \ln y_1^* \rightarrow 1/y_1^* < 1 \rightarrow y_1^* > 1$.

So it is $x < y_1^*$ if $y_1^* > 1$. But since $z = x^y$ is increasing and it's $z(0) = 1$, the first intersection of the exponential with the line $z = y$ must have a value $z > 1$. This implies (since $y = z$) that $y > 1$. So it is $y_1^* > 1$ and $x < y_1^*$. The sequence converges to y_1^* .

To complete our analysis let's see what happens with $0 < x < 1$. In this case the exponential curve $z = x^y$ is decreasing and there can be only one single intersection point with the line $z = y$ and a corresponding single fixed point. Anyway some interesting unexpected things are going to happen when we start analyzing the stability of that fixed point and the eventual convergence of the sequence to it.

Two different cobweb iteration are presented for this case in the following figures, producing rather different outcomes. If the first derivative is $|z'(y_1^*)| < 1$ (that is $-1 \leq z'(y_1^*) < 0$, Fig. 11) the iterations converge to y_1^* , oscillating between values alternatively greater and less than that of the fixed point. We can say that the fixed point is attractive and that the sequence will eventually converge to it, whatever is the starting point.

On the contrary, if the first derivative is $|z'(y_1^*)| > 1$ (that is $z'(y_1^*) < -1$, Fig. 12) the iterations

are again oscillating but the sequence doesn't converge to y_1^* . Instead it stabilizes towards a periodic stable cycle, getting closer and closer to two alternate distinct fixed values.

Let's then see for what value of x we have $z'(y_1^*) > -1$. Since we have already found that $z'(y_1^*) = x^{y_1^*} \ln x$ we must solve the inequality $x^{y_1^*} \ln x > -1$ with $y_1^* = x^{y_1^*}$ meaning $x = y_1^{1/y_1^*}$. It will then be

$$y \ln y^{1/y} > -1 \rightarrow \ln y > -1 \rightarrow y > e^{-1} \rightarrow x = y^{1/y} > e^{-e}$$

We can then say that the fixed point is attractive for $e^{-e} \leq x < 1$ and that we'll have a 2-cycle for $0 < x < e^{-e}$.

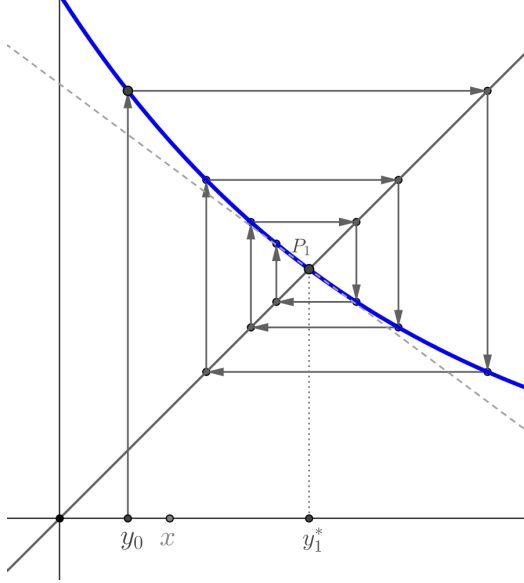


Fig. 11: Cobweb iterations in the case $0 < x < 1$ with $-1 \leq z'(y_0) < 0$ and convergence to the fixed point.

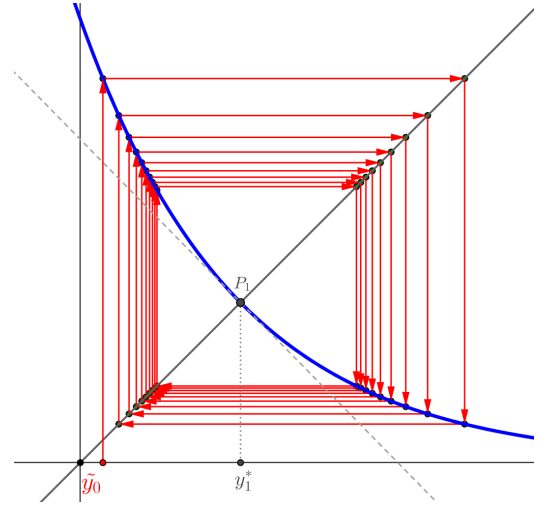


Fig. 12: Cobweb iterations in the case $0 < x < 1$ with $z'(y_0) < -1$.

What can we say about the two values y_1 and y_2 involved in the 2-cycle?

Since y_2 is the next value in the sequence after y_1 and y_1 is the next value in the sequence after y_2 we have $y_2 = x^{y_1}$ and $y_1 = x^{y_2}$. Let's take the power y_1 of both sides of the second equation to get $y_1^{y_1} = x^{y_2 y_1}$.

Inserting $x^{y_1} = y_2$ we have $y_1^{y_1} = y_2^{y_2}$ and $y_1 \ln y_1 = y_2 \ln y_2$.

Let's now say that y_2 is p times y_1 , that is $y_2 = p y_1$ and solve for y_1 .

$$p y_1 \ln p y_1 = y_1 \ln y_1$$

$$p(\ln p + \ln y_1) = \ln y_1 \rightarrow (p-1) \ln y_1 = -p \ln(p) \rightarrow \ln y_1 = p \frac{\ln p}{1-p} \rightarrow \ln y_1 = \ln \left(p^{\frac{p}{1-p}} \right)$$

We finally have

$$y_1 = p^{\frac{p}{1-p}} \quad \text{and} \quad y_2 = p y_1 = p^{\frac{p}{1-p}+1} = p^{\frac{1}{1-p}}.$$

For instance, if we set $p = 2$ we have $y_1 = 1/4$ and $y_2 = 1/2$. These are the two values of the cycle that we'd get with $x = y_2^{1/y_1} = y_1^{1/y_2} = (1/2)^4 = (1/4)^2 = 1/16 = 0.0625$. We can also find

the two values for the *extreme* cycle in which y_1 and y_2 have the maximum separation. Setting $p \rightarrow \infty$ we have

$$y_1 = \lim_{p \rightarrow \infty} p^{\frac{p}{1-p}} \rightarrow \infty^{-1} = 0^+$$

$$y_2 = \lim_{p \rightarrow \infty} p^{\frac{1}{1-p}} = \lim_{p \rightarrow \infty} e^{\frac{1}{1-p} \ln p} \rightarrow e^{0^-} = 1^-$$

and, since $y_1 = x^{y_2}$, $0 = x^1$, $x \rightarrow 0$.

The following table summarizes the results gained so far.

Values of x	Fixed point values	Fixed point(s)	Asymptotic behavior
$x > e^{1/e}$	//	No fixed points	Divergence to $+\infty$
$x = e^{1/e}$	$y = e$	1 fixed point	Convergence to the f.p.
$1 < x < e^{1/e}$	$1 < y < e$	2 fixed points	Convergence to the first f.p.
$x = 1$	$y = 1$	1 fixed point	Instantaneous convergence to the f.p.
$e^{-e} < x < 1$	$1/e < y < e$	1 fixed point	Convergence to the f.p. (oscillating)
$x = e^{-e}$	$y = 1/e$	1 fixed points	Convergence to the f.p. (oscillating)
$0 < x < e^{-e}$	$0 < y_1 < 1/e < y_2 < 1$	1 fixed point (unstable)+ a stable 2-cycle with values y_1 and y_2	Convergence to the 2-cycle
$x \rightarrow 0^+$	$y_1 \rightarrow 0$ $y_2 \rightarrow 1$	1 fixed point (unstable)+ a stable 2-cycle	Convergence to the 2-cycle
Decimal values: $e \approx 2.71828$ $e^{1/e} \approx 1.44467$ $1/e \approx 0.367879$ $e^{-e} = 1/e^e \approx 0.065988$			

Table 1: Convergence intervals of the infinite power tower.

It's interesting to note that how the number e appears in above table in many possible power variations.

In conclusion, we can now say that the infinite power tower converges to the function defined by the expression $y = x^y$ (or $x = y^{1/y}$) for $e^{-e} \leq x \leq e^{1/e}$ assuming values $1/e \leq y \leq e$.

Taking into account the information collected we can show, in Fig. 13, the final plot of the infinite power tower function.

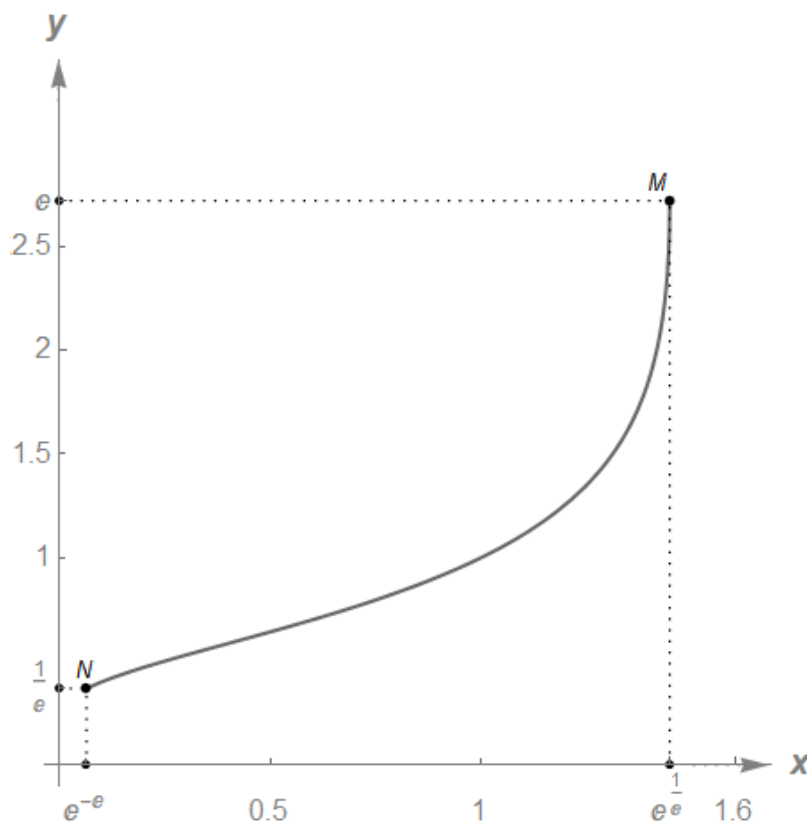


Fig. 13: Plot of $y = f(x) = x^{x^{x^{x^{...}}}}$

8 Outside the convergence interval

We have seen that the infinite power tower converges for $e^{-e} \leq x \leq e^{1/e}$, assuming values $1/e \leq y \leq e$.

But what happens outside the convergence interval?

Let's try some numerical experiment with some power towers with finite (but rather high) height.

For $x > e^{1/e}$ the function $f(x)$ blows out rapidly to $+\infty$ (Fig. 14). In fact we already know that there aren't fixed points for the sequence $y_{n+1} = x^{y_n}$ when we use $x > e^{1/e}$.

As we have already seen, for $0 < x < e^{-e}$ the sequence starts to oscillate between two bounded values, and some numerical simulation confirms that behavior (Fig. 15). The upper/lower branches of the plot correspond to an even/odd value for the height of the tower.

We have already seen this oscillating behavior when exploring, through the cobweb diagrams, the recursive sequence $y_{n+1} = x^{y_n}$ with $x < e^{-e}$.

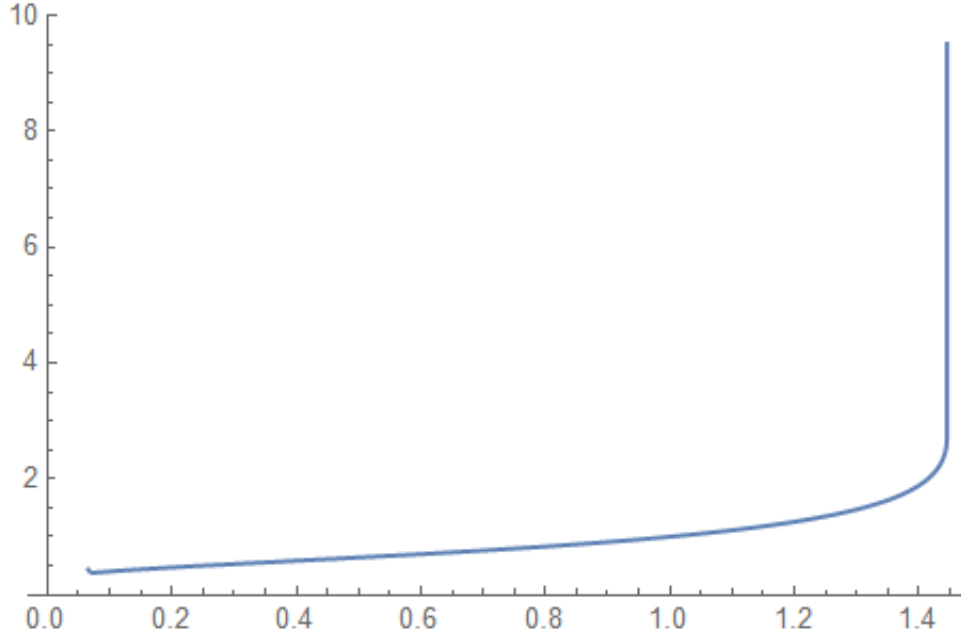


Fig. 14: Plot of $y_{200}(x) = \text{PowerTower}[x, 200]$

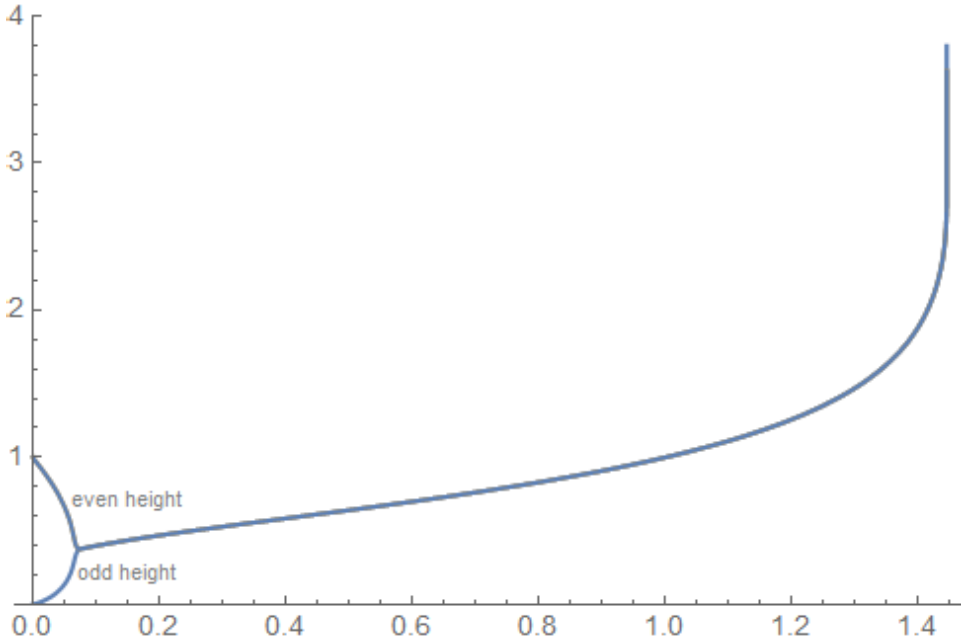


Fig. 15: Plot of $y_{200}(x) = \text{PowerTower}[x, 200]$, $y_{201}(x) = \text{PowerTower}[x, 201]$

Let's analyze further the origin of this feature.

First we can try to calculate the limit of the finite power tower when $x \rightarrow 0$. Let's start with f_2 :

$$\lim_{x \rightarrow 0} f_2(x) = \lim_{x \rightarrow 0} x^x = \lim_{x \rightarrow 0} e^{x \ln x}$$

we can now calculate the limit of the exponent (using the L'Hôpital's rule)

$$\lim_{x \rightarrow 0} x \ln x = \lim_{x \rightarrow 0} \frac{\ln x}{1/x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0} (-x) = 0^-$$

so it is

$$\lim_{x \rightarrow 0} e^{x \ln x} = 1^-$$

Then we are on the upper branch. What changes when we increase the tower height?

$$\lim_{x \rightarrow 0} f_3(x) = \lim_{x \rightarrow 0} x^{x^x} = \lim_{x \rightarrow 0} x^{(x^x)}$$

and since $\lim_{x \rightarrow 0} x^x = 1$, as seen before, the last limit has the form $0^1 \rightarrow 0$

So it is $\lim_{x \rightarrow 0} f_2(x) = \lim_{x \rightarrow 0} x^x = 1$ and $\lim_{x \rightarrow 0} f_3(x) = \lim_{x \rightarrow 0} x^{x^x} = 0$.

We have a strong suspect (supported by the previous reasoning based on the cobweb diagrams) that these results may extend to towers with any height, with different values for n even and n odd, that is $\lim_{x \rightarrow 0} f_{2n}(x) = 1$ and $\lim_{x \rightarrow 0} f_{2n+1}(x) = 0$ but we can't prove this conjecture with simple tools and leave this problem to a later time.

Having observed the oscillating character of the finite power tower sequence for $0 < x < e^{-e}$, we ask ourselves if it's possible to find the equations of the two distinct branches.

Calling a and b the two values corresponding to some \bar{x} it must be:

$$y_n(\bar{x}) \rightarrow a, \quad y_{n+1}(\bar{x}) \rightarrow b, \quad y_{n+2}(\bar{x}) \rightarrow a, \quad y_{n+3}(\bar{x}) \rightarrow b, \dots$$

This means that **the sequence built with a double recursion** should converge to its fixed points a and b .

Let's see what is the form of this double recursion:

$$\begin{cases} y_{n+1} = x^{y_n} \\ y_{n+2} = x^{y_{n+1}} \end{cases} \Rightarrow y_{n+2} = x^{x^{y_n}}$$

This sequence has stable fixed points if the derivative with respect to y of the right side has modulus less than 1, that is

$$\left| \frac{d}{dy} (x^{x^y}) \right| < 1$$

Since it is $x^{x^y} = e^{\ln x^{x^y}} = e^{x^y \ln x}$ the derivative to calculate becomes:

$$\frac{d}{dy} (e^{x^y \ln x})$$

after some passage we arrive at

$$\frac{d}{dy} (e^{x^y \ln x}) = x^{x^y+y} \ln^2 x$$

and it must be

$$|x^{x^y+y} \ln^2 x| < 1$$

Differently from before we can't find an explicit algebraic form for the boundary of the region of convergence.

Anyway, using the *RegionPlot* command of *Mathematica* we can visualize it (Fig. 16). The double step sequence converges in the gray region and does not in the white one.

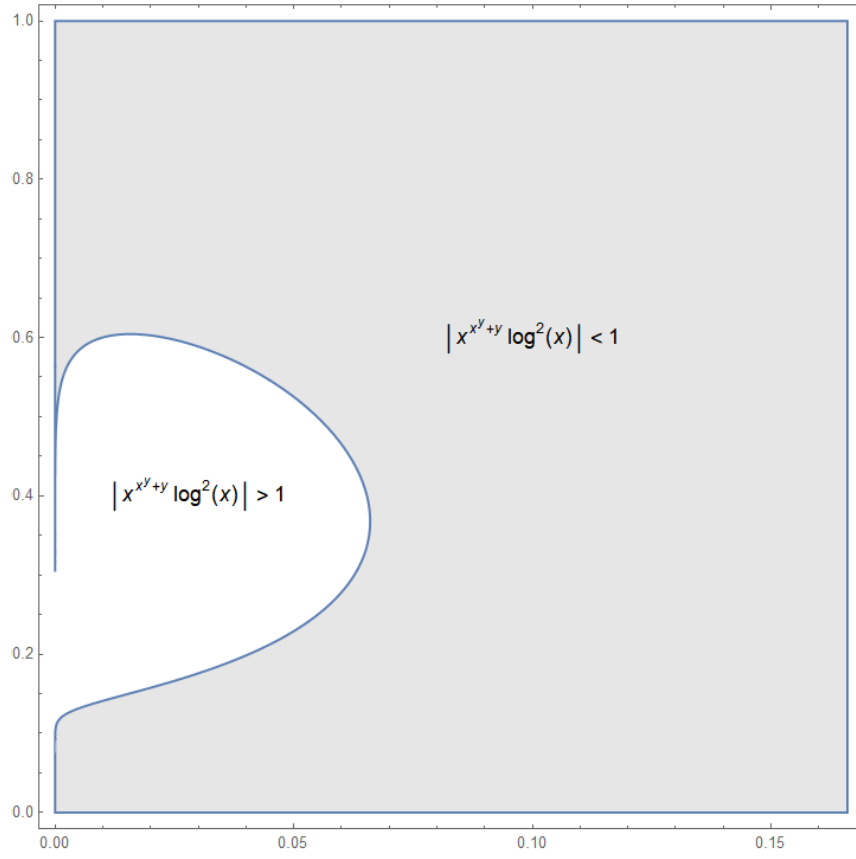


Fig. 16: RegionPlot of $|x^{x^y+y} \ln^2 x| < 1$

In the gray region, where the double step sequence converge, it will converge to the sequence whose fixed points are given by the transcendental equation

$$y = x^{x^y}$$

Now we can now put all the pieces together.

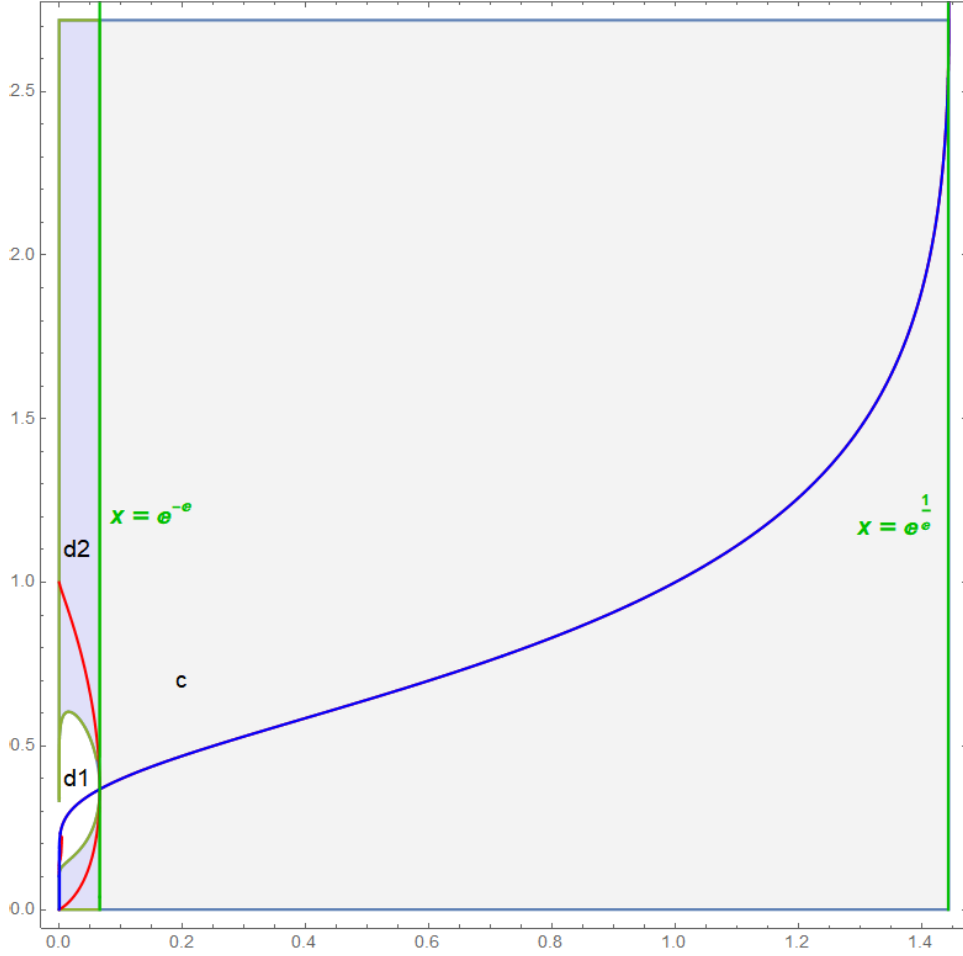


Fig. 17: All the plots together: $y = x^y$ and $y = x^{x^y}$

In Fig. 17 we can see the plots p_1 and p_2 defined by the equations $y = x^y$ and $y = x^{x^y}$ respectively. These equations are also the equations defining the fixed points of the sequences $s_1 : y_{n+1} = x^{y_n}$ and $s_2 : y_{n+2} = x^{x^{y_n}}$. The gray region is where both sequences converge (“c” in the figure), while the white area (“d1” in the figure) is a region where there’s no convergence. The blue line is produced by both equations (since the fixed points of a “single iteration” sequence are also fixed points for the one with a “double iteration” step). At the left of the line $x = e^{-e}$ (“d2” in the figure) there’s no convergence for s_1 and we have three branches. The upper and lower branches (in red) are produced only by the equation $y = x^{x^y}$.

Since they lie in a region of convergence for this sequence their values can be also produced by the infinite power tower function and we’ll have alternating values, one on the upper branch (for even heights of the tower) and the other on the lower branch (odd values of the heights). The middle branch represent points produced by both equations. Anyway this branch is entirely located in the region “d1” where there is no convergence for both s_1 and s_2 . The infinite power tower won’t assume these values.

Fig. 18 shows an enlargement of the region with the three branches.

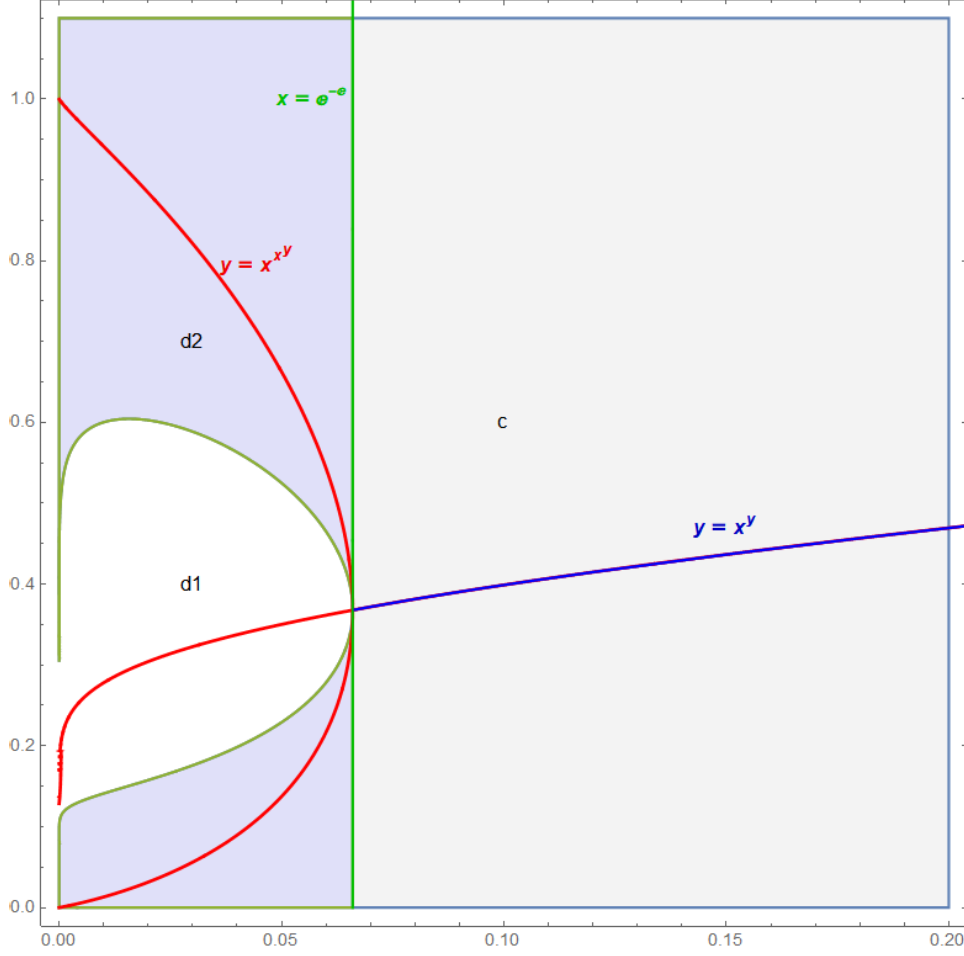


Fig. 18: Detail of $y = x^y$ and $y = x^{x^y}$ near the “pitchfork bifurcation”

Now we can find an answer to our previously unanswered question: what is the limit of the infinite power tower function when $x \rightarrow 0$?

The answer is: that limit doesn't exist. In fact, more precisely, we can have two distinct values for that limit.

That's because, since in the converging region it is $y = x^{x^y}$, we have

$$\begin{aligned} \text{for } y \rightarrow 1 \quad & \lim_{\substack{y \rightarrow 1 \\ x \rightarrow 0}} x^{x^y} = \lim_{x \rightarrow 0} x^x = \lim_{x \rightarrow 0} e^{x \ln x} = 1 \\ \text{for } y \rightarrow 0 \quad & \lim_{\substack{y \rightarrow 0 \\ x \rightarrow 0}} x^{x^y} = \lim_{x \rightarrow 0} x^1 = \lim_{x \rightarrow 0} x = 0 \end{aligned}$$

and the equation $y = x^{x^y}$ is verified for both $\begin{cases} x \rightarrow 0 \\ y \rightarrow 1 \end{cases}$ and $\begin{cases} x \rightarrow 0 \\ y \rightarrow 0 \end{cases}$

The emergence of the transition from a single fixed point to a 2-cycle can be better understood by seeing how the function $y_{n+2} = x^{y_n}$ changes with different values of the x . Again, let's use $z_2 = y_{n+2}$ and $y = y_n$

For $e^{-e} \leq x \leq e^{1/e}$ our double step function $z_2 = x^{x^y}$ (solid line in the figures) has the same fixed points of $z = x^y$ (dashed line). Anyway, when $x < e^{-e}$, two new intersections with the identity

line appear Fig. (22). They correspond to the values of the stable 2-cycle. In the meantime, the fixed point y_1^* change from attractive to repulsive. In the theory of dynamical systems the transition from one fixed point to three fixed points is called ***pitchfork bifurcation***.

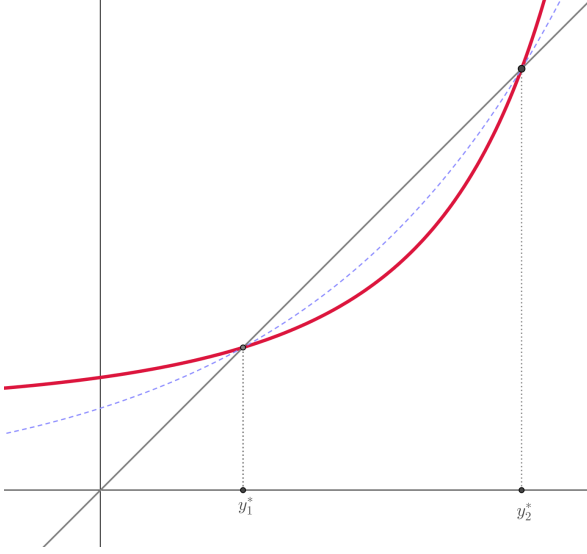


Fig. 19: Graph of $z_2 = x^{x^y}$ for $1 < x < e^{1/e}$

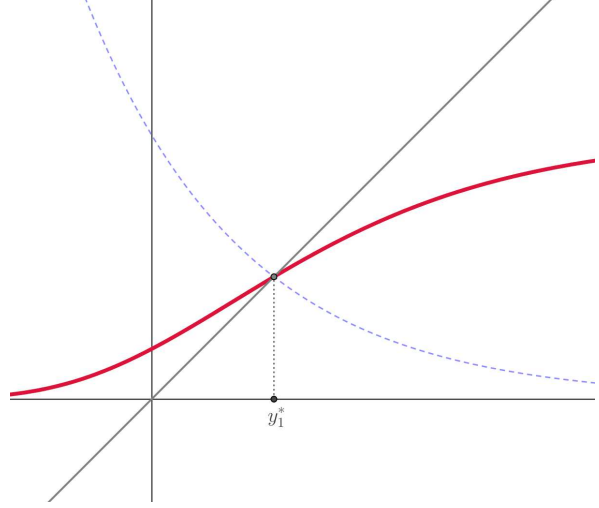


Fig. 20: Graph of $z = x^{x^y}$ for $e^{-e} < x < 1$.

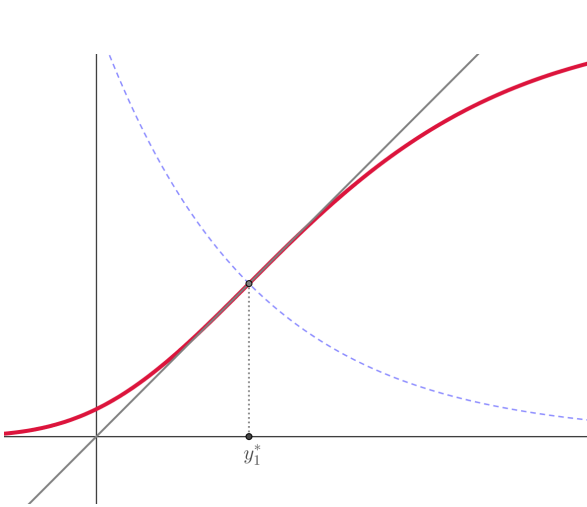


Fig. 21: Graph of $z = x^{x^y}$ for $x = e^{-e}$.

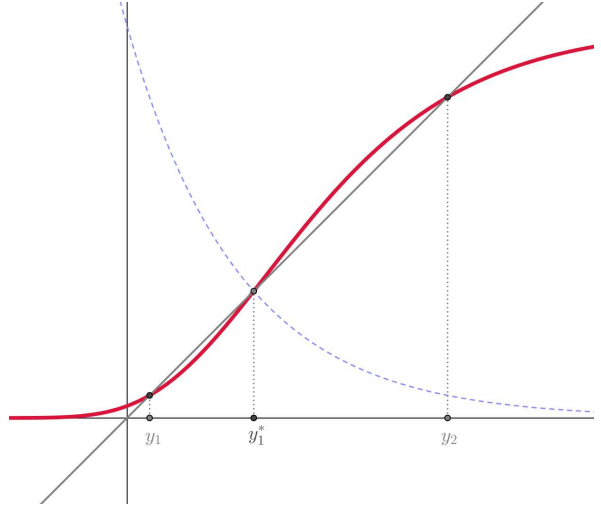


Fig. 22: Graph of $z = x^{x^y}$ for $0 < x < e^{-e}$.

9 Some history about the power tower

What is the origin of the power tower function? How come that someone had the idea of creating such a *monster*? Actually its genesis can, somehow, be connected with the arithmetical operations based on Peano's axioms³:

1. zero is a number.
2. if a is a number, the successor $S(a)$ of a is a number.
3. zero is not the successor of a number.
4. two numbers of which the successors are equal are themselves equal.
5. If a set K of numbers contains zero and also the successor of every number in K , then every number is in K (induction axiom).

Peano's axioms are the basis of the arithmetic of natural numbers, where the operations of addition, multiplication and exponentiation can be defined. Yet the only (unary) operation included in Peano's axiom is the successor.

However, we can build the other operations by iterating the one defined at the previous step. The operations defined in this way are called *hyperoperations*, and the grade 0 of this sequence is the *successor* operation that, if iterated, can be used to define any natural number.

So we can build the sequence of operations shown in the following table:

	Name	Definition
hyper0	Successor	$S(n) = n + 1$
hyper1	Addition	$n + m = S^m(n)$
hyper2	Multiplication	$n \cdot m = \underbrace{n + n + n + \dots + n}_m$
hyper3	Exponentiation	$n^m = \underbrace{n \cdot n \cdot n \cdot \dots \cdot n}_m$
hyper4	Tetration	${}^m n = \underbrace{n^{n^{\dots^n}}}_m$

The sequence of hyperoperations can go on with the hyper5 (pentation), the hyper6 (hexation) and beyond.

Naturally, the commonly used operations are the ones reaching hyper3 (exponentiation), but we can see that the *tetration* is not just an exotic oddity but can be thought of as an extension of the process leading to the most usual arithmetical operations.

The tetration with infinite height (infinite power tower) is often dealt together with the *Lambert W* function (called `ProductLog` in *Mathematica* and `LambertW` in *Geogebra*).

The Lambert W function $y = W(x)$ is defined as the inverse function of $x = y \cdot e^y$ (note that there's no algebraic closed form expression for this function).

³<http://mathworld.wolfram.com/PeanosAxioms.html>

The LambertW function can be used to solve certain type of transcendental equations such as, for instance, $x e^x = 2$. Its solution can be written as $x = \text{LambertW}(2)$ and the numerical value returned is 0.852606 (since $0.852606 \cdot e^{0.852606} = 2$).

Taking advantage of the definition of the LambertW function, the fixed points of the infinite power tower can be expressed as

$$y = x^y = \frac{W(-\ln x)}{-\ln x}$$

In fact, starting from

$$y = x^y \rightarrow y = e^{y \ln x} \rightarrow y e^{-y \ln x} = 1, \text{ multiply both sides by } -\ln x$$

$$-y \ln x \cdot e^{-y \ln x} = -\ln x$$

set $w = -y \ln x$; $z = -\ln x$

$w e^w = z \rightarrow w = W(z)$ that is, by definition, the LambertW function.

Substitute back the w and z

$-y \ln x = W(-\ln x) \rightarrow y = \frac{W(-\ln x)}{-\ln x}$ that is the explicit form of the implicit function defined by $y = x^y$



Johann Heinrich Lambert
(1728-1777)

The definition of the Lambert W function originated by the article “*Observationes variae in mathesin puram*”⁴ published in 1758 by the Swiss mathematician **Johann Heinrich Lambert** in which he dealt with the solution of the trinomial transcendental equation $x^m + px = q$ and discovered that, under certain conditions, the solution (a solution) could be expressed with the following series:

$$x = \frac{q}{p} - \frac{q^m}{p^{m+1}} + m \frac{q^{2m-1}}{p^{2m+1}} - m \frac{3m-1}{2} \frac{q^{3m-2}}{p^{3m+1}} + m \frac{4m-1}{2} \frac{4m-2}{3} \frac{q^{4m-3}}{p^{4m+1}} - m \frac{5m-1}{2} \frac{5m-2}{3} \frac{5m-3}{4} \frac{q^{5m-4}}{p^{5m+1}}$$

To derive above series Lambert used a procedure that was later generalized by **Joseph-Louis Lagrange** in 1770⁵ with what’s presently known as “*Lagrange inversion theorem*”.

With Lagrange’s method, given a polynomial function⁶ $y = f(x) = a_1x + a_2x^2 + \dots + a_mx^m$ it’s possible to find the series expansion of the inverse function $x = g(y) = A_1y + A_2y^2 + \dots$ by applying the following steps⁷:

- plug the first expression in the second

$$x = A_1 (a_1x + a_2x^2 + \dots + a_mx^m) + A_2 (a_1x + a_2x^2 + \dots + a_mx^m)^2 + A_3 (a_1x + a_2x^2 + \dots + a_mx^m)^3 + \dots$$

- equate the coefficients of the right and left sides having the same grade of the x .

$$\begin{aligned} A_1 a_1 &= 1 & \rightarrow & A_1 = \frac{1}{a_1} \\ A_1 a_2 + A_2 a_1^2 &= 0 & \rightarrow & A_2 = -\frac{a_2}{a_1^3} \\ A_1 a_3 + 2A_2 a_1 a_2 + A_3 a_1^3 &= 0 & \rightarrow & A_3 = \frac{2a_2^2 - a_1 a_3}{a_1^5} \\ &\dots & & \dots \end{aligned}$$

⁴Lambert J. H., (1758). *Observationes variae in mathesin puram*, Acta Helveticae physico-mathematico-anatomico-botanico-medica, Band III, 128–168, 1758.

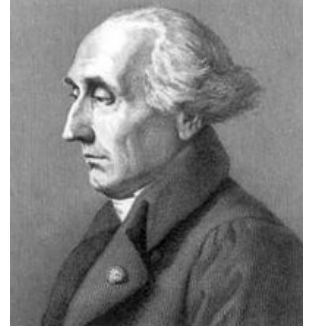
⁵Lagrange, Joseph-Louis (1770). *Nouvelle méthode pour résoudre les équations littérales par le moyen des séries*, Mémoires de l’Académie Royale des Sciences et Belles-Lettres de Berlin. 24: 251–326.

⁶If the polynomial contains a constant term a_0 it’s possible to eliminate it with a change of variable $y \mapsto y - a_0$

⁷<http://mathworld.wolfram.com/SeriesReversion.html>

By finding the inverse function (or, better, an approximation of the inverse function around the point $x_0 = 0$) it is also possible to find the approximate value of a root \tilde{x} of a polynomial equation having the form $f(x) = q$ since it is $\tilde{x} = g(f(\tilde{x})) = g(q)$.

This procedure can be extended to generic (not polynomial) functions $z = f(w) \rightarrow w = g(z)$ using a more general form of the Lagrange inversion theorem. Naturally there is the problem of convergence of the series, problem that we won't discuss here.



Joseph-Louis Lagrange
(1736-1813)

In a subsequent article, “*Observations analytiques*”⁸ published in 1772, Lambert, examined the similar trinomial equation $x = q + x^m$ and wrote down the series that express not only a root of the equation, but also the powers of that root. In this article Lambert also mentions his meeting with L. Euler in Berlin in 1764 and their discussions about the series connected with polynomial equations.

Some years later, in 1779, **Leonhard Euler** published “*De serie Lambertina plurimisque eius insignibus proprietaribus*”⁹ in which, referring to the previous works by Lambert, he investigated the solutions of another trinomial equation, equivalent to the one studied by Lambert, having the form

$$x^\alpha - x^\beta = v(\alpha - \beta)x^{\alpha+\beta}$$

The equivalence can be verified by choosing the transformation of the parameters $\alpha = -m$, $\beta = -1$, $v(\alpha - \beta) = q$, obtaining

$$\frac{1}{x^m} - \frac{1}{x} = \frac{v(\alpha - \beta)}{x^{m+1}} \rightarrow x - x^m = q$$

In this case the series useful to express the solution (or one of its powers) is¹⁰:

$$x^n = 1 + nv + \frac{1}{2}n(n + \alpha + \beta)v^2 + \frac{1}{6}n(n + \alpha + 2\beta)(n + 2\alpha + \beta)v^3 + \frac{1}{24}n(n + \alpha + 3\beta)(n + 2\alpha + 2\beta)(n + 3\alpha + \beta)v^4 + \dots$$

Euler then makes a transformation of both expressions in the special cases $\alpha \rightarrow \beta \rightarrow 1$ for the first equation and $\alpha \rightarrow \beta \rightarrow 1 \wedge n \rightarrow 0$ for the second.

For the first expression ($x^\alpha - x^\beta = v(\alpha - \beta)x^{\alpha+\beta}$) it is

$$\frac{x^\alpha - x^\beta}{\alpha - \beta} = vx^{\alpha+\beta} \rightarrow \frac{x^\alpha(1 - x^{\beta-\alpha})}{\alpha - \beta} = vx^{\alpha+\beta} \rightarrow \frac{x^\alpha(x^{\beta-\alpha} - 1)}{\beta - \alpha} = vx^{\alpha+\beta}$$

and taking the limit $\beta \rightarrow \alpha$, $\beta - \alpha \rightarrow \varepsilon$, $\varepsilon \rightarrow 0$

$$\lim_{\varepsilon \rightarrow 0} \frac{x^\alpha(x^\varepsilon - 1)}{\varepsilon} = vx^{\alpha+\alpha} \Rightarrow x^\alpha \ln x = vx^{2\alpha} \Rightarrow \ln x = vx^\alpha$$

⁸Lambert J. H., (1770). *Observations analytiques*, Nouveaux Mémoires de l'Académie royale des sciences de Berlin, année 1770/1772

⁹Euler L., (1779). *De serie Lambertina plurimisque eius insignibus proprietaribus*, originally published in “Acta Academiae Scientiarum Imperialis Petropolitinae” 1779, 1783, pp. 29-51

¹⁰Euler's series is not equivalent to Lambert's because Euler's series is centered around 1 and Lambert's is centered around 0.

and, since $\alpha \rightarrow 1$, $\ln x = vx$.

For the second expression it is

$$\frac{x^n - 1}{n} = v + \frac{1}{2}(n + \alpha + \beta)v^2 + \frac{1}{6}(n + \alpha + 2\beta)(n + 2\alpha + \beta)v^3 + \frac{1}{24}(n + \alpha + 3\beta)(n + 2\alpha + 2\beta)(n + 3\alpha + \beta)v^4 + \dots$$

and taking the limits for $\alpha \rightarrow 1$, $\beta \rightarrow 1$, $n \rightarrow 0$ it is

$$\lim_{n \rightarrow 0} \frac{x^n - 1}{n} = \ln x = v + v^2 + \frac{3}{2}v^3 + \frac{8}{3}v^4 + \dots$$

Putting together the two expressions we can say that a special solution of Euler's trinomial equation can be written in two different ways:

(1) the solution of $\ln x = vx$ and

(2) the result of the series $\ln x = v + v^2 + \frac{3}{2}v^3 + \frac{8}{3}v^4 + \dots$

This means that the solution of the transcendental equation $\ln x = vx$ can be expressed by the series $\ln x = v + v^2 + \frac{3}{2}v^3 + \frac{8}{3}v^4 + \dots$ and if we set $\ln x = t$ (and $x = e^t$) we have $t = ve^t$ whose solution is $t = v + v^2 + \frac{3}{2}v^3 + \frac{8}{3}v^4 + \dots$

The equation $t = ve^t$ can be rewritten as $-te^{-t} = -v$ and, using the definition of the Lambert W function as solution of $we^w = z \Rightarrow w = W(z)$ we have $-t = W(-v)$ that is $t = -W(-v)$. So here we have the series expansion

$$-W(-v) = v + v^2 + \frac{3}{2}v^3 + \frac{8}{3}v^4 + \dots \text{ and } W(-v) = -v - v^2 - \frac{3}{2}v^3 - \frac{8}{3}v^4 + \dots$$

that is (setting $-v = z$),

$$W(z) = z - z^2 + \frac{3}{2}z^3 - \frac{8}{3}z^4 + \dots$$

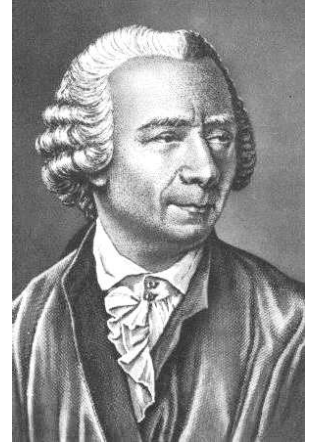
representing the series expansion of the Lambert W function.

Even more closely related with the subject of this article is another work by Euler: "*De formulis exponentialibus replicatis*"¹¹, presented in 1777 (two years before the publication of "*De serie Lambertina*"), in which he investigated a problem posed by the French philosopher and mathematician Nicolas de Condorcet (known as Marquis de Condorcet), regarding the convergence of the sequence $r, r^\alpha, r^{r^\alpha}, r^{r^{r^\alpha}}, \dots$

The article's opening is very interesting to point out Euler's keen interest in such expressions. Its translation goes more or less like this:

"The famous Marquis de Condorcet recently shared with the academy deep speculations regarding some rather unfamiliar analytic formulas, among which we can, first of all, include the formulas called repeated exponentiations, where every power goes into the power exponent following it. Yet, little has been achieved about the nature of such expressions and despite the force of those investigations, led with incredible sagacity, no clear knowledge and perception has been reached. Hence it will not be useless to explain here some special properties of such expressions."

¹¹Euler L., (1777). *De formulis exponentialibus replicatis*, presented to the St. Petersburg Academy in 1777 and published in "Acta Academiae Scientiarum Imperialis Petropolitinae 1, 1778". Also in Opera Omnia: Series 1, Volume 15, pp. 268 – 297.



Leonhard Euler
(1707-1783))

In the article Euler proves that the sequence $r, r^\alpha, r^{r^\alpha}, r^{r^{r^\alpha}}, \dots$ converges if $e^{-e} < r < e^{1/e}$.

He also notes (p. 57) that the sequence $\beta = r^\alpha, \gamma = r^\beta = r^{r^\alpha}, \delta = r^\gamma = r^{r^{r^\alpha}}, \dots$ may produce an alternate sequence of two values. In fact, choosing $r = 1/16$ and $\alpha = 1/2$ we have

$$\beta = r^\alpha = (1/16)^{1/2} = 1/4, \gamma = r^\beta = (1/16)^{1/4} = 1/2, \delta = r^\gamma = (1/16)^{1/2} = 1/4$$

and so on, with the results assuming the alternating values $1/2$ and $1/4$. Euler shows that, in general, this happens when $r^\Phi = \Psi$ and $r^\Psi = \Phi$ leading to the identity $\Phi^\Phi = \Psi^\Psi$, since it is

$$r^{\Phi \cdot \Psi} = \Psi^\Psi \rightarrow (r^\Psi)^\Phi = \Psi^\Psi \rightarrow \Phi^\Phi = \Psi^\Psi$$

Now, this equation doesn't necessarily imply that $\Phi = \Psi$, because the function $y = x^x$ has a turning point for $x = 1/e$ and some y can be obtained with two different values of the x .

To find the relation between the two values satisfying the equation Euler sets $\Psi = p \cdot \Phi$ and finds that it must be¹²

$$\Phi = p^{p/(1-p)}, \Psi = p^{1/(1-p)} \text{ and } r = \Phi^{1/\Psi} (= \Psi^{1/\Phi})$$

Finally Euler asks himself which is the condition for this two values to converge to a single value. This happens for $p = 1$ and it is $\Phi = \lim_{p \rightarrow 1} p^{p/(1-p)} = e^{-1}, \Psi = \lim_{p \rightarrow 1} p^{1/(1-p)} = e^{-1}$

The corresponding value of r is

$$r = \Psi^{1/\Phi} = (1/e)^e = e^{-e}$$

He then concludes that the relations $r^\Phi = \Psi$ and $r^\Psi = \Phi$ will always yield two different values if $r < e^{-e}$.

10 Conclusions

We have considered the function based on a reiterated exponentiation $y = x^{x^{x^{\ddots}}}$ and have investigated its properties, finding some counterintuitive fact. During our journey we had to cope with the unusual definition of this function, with its infinite sequence of exponents piling up one over the others. To proceed forward and make some headway we had to use different mathematical arguments, such as the concept of function and inverse function, limits and derivatives, exponentials and logarithms, sequences, fixed points of recursive sequences, cobweb diagrams and others. We also used experimental empirical tools like complex numerical computations and graphical plots provided by mathematical software packages. At the end we can say that much of the properties characterizing the infinite power tower function and its convergence (or not) to finite values have been explained.

Anyway, what we are left with is a vague sense of awe and amazement in observing the mysterious metamorphosis of this function, from one leading to infinite results (as it was expected in the very early stages, before starting a more in-depth analysis), to one producing finite values and, lastly, to one undergoing some serious structural change (called bifurcation in the field of dynamical systems) and generating stable 2-cycles.

¹²We have used Euler's method in Section 7.

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