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INTRODUCTION

Darboux's greatest love

1. Introduction

This paper aims to answer the following question: what are the origins of Darboux's lifelong passion for the geometric subject of orthogonal coordinates? It also serves as a brief introduction to the fundamental notions of old algebraic geometry.

Studying the writings of the old Masters of geometry is laborious, and yet it is always a good intellectual investment. In fact, it is an excellent illustration of the Latin proverb *Per aspera ad astra*, i.e. 'Through difficulties to the stars'.

Jean Gaston Darboux (b. 14 August 1842, d. 23 February 1917) wrote his first paper while he was still a student at the famous l'École Normale Supérieure [1]. The paper dealt with the plane sections of the torus. A standard torus is an algebraic surface of fourth order: any straight line meets a torus at four points (real or imaginary). Thus, any toric section is an algebraic curve again of fourth order. The variety of the toric sections is quite rich. The study of the case when sectioning planes are parallel to the symmetry axis of the torus dates back to antiquity. The corresponding curves are usually called 'the spiric sections of Perseus' after their discoverer (circa 150 BC). Among them, one can distinguish Cassini's curves (in particular, Bernoulli's lemniscate). The other (i.e. oblique) particular toric sections are the famous Villarceau circles [2]. The update of Villarceau's discovery is as follows: a complete, sufficiently smooth surface with the property that through each point on the surface exactly four circles of the surface traverse is a standard torus and vice versa [3]. This result proves that the subject is not elementary. Indeed, the summary of the first paper by Darboux has been given by Eisenhart as follows [4]:

> It was shown that the curve (toric section) is of the fourth order with the circular points at infinity for double points; that there are sixteen foci, in the sense of Plücker, of which four are real and lie on a circle; that there is a homogeneous linear relation between the distances of any point of the curve from three of these foci; and that the inversion w.r.t. any of these foci as pole transforms the curve into an oval of Descartes.

Presumably, even for present-day specialists in algebraic geometry, the text is not fully accessible: some topics of old algebraic geometry are now out of fashion. In section 5, I shall try to 'decode' the above-mentioned statements by Eisenhart.

The results of the 22 year old student impressed many contemporaries; among them was Michel Chasles, a great authority on the geometry of that period. It was Chasles who began a modern treatment of both toric sections and Cartesian ovals; see his very influential *Aperçu historique* [5]. Chasles agreed to be a supervisor of Darboux's doctorate. On 14 July 1866 (under Napoleon III, it was not a national holiday!) in the presence of Chasles, Joseph-Alfred Serret and Jean Claude Bouquet, Darboux defended his doctoral dissertation published as *Recherches sur les Surfaces Orthogonales* [6] and *Sur les Surfaces Orthogonales* [7]. The latter is also the title of the thesis.

At first sight, there is no connection between the first work and the thesis. This impression is wrong. Darboux as a student wrote the second paper [8] and Eisenhart, discussing their importance, remarks:

> . . . they contain germs of his subsequent work.

'Subsequent work' means all his papers were devoted explicitly or implicitly to orthogonal coordinates on a plane or in a space.

Having completed his doctoral studies, from 1 October 1867 to 26 September 1872 Darboux taught in the famous French secondary school Lycée Louis-le-Grand. Nevertheless, his scientific ties with Paris academicians remained. In spite of the dramatic historical events of by the Franco-Prussian War (15 July 1870–1 February 1871) and the Paris Commune (18 March 1871–28 May 1871), this period was the most productive in Darboux's scientific career. In particular, he was able to formulate precisely the program of geometric research for the rest of his life. His manifesto is declared in his book [9], published in 1873. We know, however, that Darboux completed the manuscript before 7 June 1869 [10]. At that time, printing was impossible due to the aforementioned historical events.

The title of the book summarizes its content. The central themes are 'cycliques' and 'cyclides'. Cycliques are special algebraic curves of at most fourth order which can be defined either by description of their singularities or analytically. Particular cycliques are conics, Cartesian ovals and toric sections. Direct two-dimensional analogues of cycliques are called cyclides since a particular class of cyclides are the famous Dupin cyclides. The point is that both cycliques and cyclides can be used to construct non-trivial orthogonal coordinates on a plane and in a space, respectively. Both terms are due to Darboux [10]. The term 'cyclide' has been generally adopted in English, but 'cyclique' is replaced by 'bicircular quartics' when it is of fourth order.

In 1872, Darboux's growing fame brought him the position of teacher at the l'Ecole Normale [11]. This was the beginning of his great career as a scientist and as a teacher. It is not easy to enumerate all functions, responsibilities and honors which came to him in his very rich life. In 1910 Ernest Lebon [12] gave a long, impressive list of *Grades.Fonctions.Titres Honorifiques.Prix.Décorations*. In particular, Darboux was a member of many academies and scientific societies. On 3 March 1884, he was elected to be a member of the very prestigious l'Académie des Sciences de Paris (Section Géométrie). Later, on 21 May 1900, he became the permanent secretary (for mathematics) of the Academy as the successor of Joseph Bertrand. With the advent of *Comptes Rendus* (the official journal of the Academy) in 1835, the role and prestige of the permanent secretaries was considerably enhanced: they were responsible for the publishing policy of the Academy.

In [11], Struik writes:

> Darboux was primarily a geometer but had the ability to use both analytic and synthetic methods . . . .

Eisenhart writes more explicitly:

> Darboux's ability was based on a rare combination of geometrical fancy and analytical power.

Darboux was a prolific and successful writer. His various writings cover the following areas: analysis, differential geometry, analytical geometry, mathematical physics, classical mechanics and science history. The final sentence of the 1910 biography [12] is as follows: 'The number of writings of M Darboux is 419'. This is certainly an underestimate: e.g. two early papers [1, 8] are not counted. Obviously, Lebon's statistics do not include the period 1910–1917. This huge diversity of innumerable ideas and results does not preclude the existence of one prevailing theme. Struik claims:

> In his *Leçons sur les Systèmes Orthogonaux* (1898), he returned to his **early love** . . . .

I would modify this opinion: the subject of orthogonal coordinates was Darboux's *greatest love* (in a purely scientific sense of this word!). Indeed, it is by no means accidental that

(i) the first two books by Darboux [9, 13] treat orthogonal coordinates as a final goal,
(ii) his last book (completed six weeks before his death) is deeply rooted in his manifesto [9],
(iii) his great monograph on orthogonal and conjugate (!) coordinates [14, 15] is still a bestseller,
(iv) in the period 1864–1909, he wrote at least 29 papers related to orthogonal systems, cycliques and cyclides; one of the highlights of his research on the whole is a great memoir on orthogonal systems [16]: the monograph [14, 15] includes much of the material of [16].

2. Questions

Jean Gaston Darboux began his studies in October 1861 first at l'École Polytechnique and finally at l'École Normale Supérieure. His unusual intellectual ability, his enthusiasm for mathematics and his rare powers of expression soon made him conspicuous. Earlier M Nisard, the director of l'École Normale Supérieure, in an official letter of 9 November 1861 addressed to *ministre de l'Instruction Publique* described Darboux as 'a young man of a rare knowledge and of high expectations'.

In the 1860s, French mathematics with Paris as its leading center still enjoyed dominance in Europe. While a student, Darboux interacted with the most eminent French mathematicians of the period. At the outset of his studies, he assisted Joseph Bertrand (1822–1900) in lecturing. It was the beginning of their long friendship. Later, the young Darboux acquired the friendship of other Paris mathematicians: Michel Chasles (1793–1880), Joseph Serret (1819–1885), Jean Bouquet (1819–1885) and Charles Briot (1817–1882). The young man was indeed fortunate! In 1861, Chasles, Bertrand and Serret were members of the First Section (Geometry) of the Academy of Sciences of Paris. The other members of the Section were Gabriel Lamé, Jean-Baptiste Biot and Charles Hermite.

For us, the points at issue are as follows.

1. What were the origins of Darboux's lifelong passion for the geometric subject of orthogonal systems?
2. What was the intellectual pathway Darboux selected to achieve his doctoral degree?

Before answering these questions, I shall outline some fundamental terms of old algebraic geometry, also called higher geometry.

3. Some terms of higher geometry [17, 18]

In what follows, I assume rudimentary knowledge of algebraic geometry in CP^2—a complex projective plane. Recall that CP^2 is composed of C^2 and a line at infinity L which is homeomorphic to a 2-sphere.

3.1. Circular points at infinity

This notion (1822) is due to Jean Victor Poncelet, the cofounder of projective geometry. Every circle in the complex projective plane intersects L at two points, say I and J. Their homogeneous coordinates are [1, i, 0] and [1, −i, 0], respectively. Hence, any two circles (also concentric) always intersect at I and J. The term (obvious) was introduced by George Salmon, one of the most influential English geometers of the nineteenth century.

3.2. Class of the algebraic curve

The principle of duality plays a central role in higher geometry. Its first formulation was given in 1826 by Joseph Diaz Gergonne, the other cofounder of projective geometry. In plane geometry, dualistic notions are points and (straight) lines. For example, any algebraic curve can be viewed either as a trace of a movable point or as an envelope of a movable line. Accordingly, any algebraic curve has two 'discrete' characteristics: degree n and class m. A curve is of class m when m tangents can be drawn to it through any point of the plane. In 1829 J V Poncelet proved that in general $m \leqslant n\,(n-1)$. The equality holds only when a curve is singularity-free. For non-degenerate conics, $n = 2$ and $m = 2$.

3.3. Algebraic curves in terms of singularities

Singularities of a given algebraic curve can be situated in various subsets of CP^2. In particular, they can be finite and real and as such they are the topics treated in standard textbooks on analytic geometry. More exotic singularities are those belonging to the line at infinity. In analytic geometry, one considers nodes and their limiting case cusps and many other singularities. All these notions can be extended to the complex projective plane. The theory, commencing with Julius Plücker's results (1842), was completed in 1866 by Arthur Cayley—the 'Victorian Leonhard Euler'. It turns out that the most interesting case of imaginary singularities at infinity occurs when the singularities are circular points at infinity.

The Plücker–Cayley formalism enables one to define an algebraic curve or, better, a class of algebraic curves in a coordinate-free way: by description (type and localization) of pertinent singularities.

3.4. Bicircular quartics

This is a rich class of algebraic curves possessing a pair of nodes at the circular points at infinity. The real affine part of bicircular quartics is given by

$$(x^2 + y^2)^2 + u_1(x^2 + y^2) + u_2 = 0.$$

Here, a letter with subscript n denotes a polynomial of nth order in x and y.

Cartesians (nodes are replaced by cusps) are a limiting case of bicircular quartics and hence the formula comprises them as well. An important example of the Cartesian is the Cartesian oval. Toric sections are bicircular quartics (or Cartesians) with an axis of symmetry. The term 'bicircular quartics' was probably coined by A Cayley in the 1850s. One can show that a bicircular quartic is of class $m = 8$, 6 and 5 while a Cartesian is of class $m = 6$, 4 and 3.

3.5. Foci of algebraic curves

Non-degenerate conics have foci. Any ellipse has four finite foci: two real (standard) and two imaginary. This statement follows from a remarkable definition of the focus of a general algebraic curve proposed by Plücker in 1833.

Consider an algebraic curve of nth order ($n > 1$) and of mth class. From the circular point at infinity I, one can draw exactly m (not necessarily distinct!) tangents to the curve : $IP_1, \ldots, IP_m$ (P_i—points of tangency) and likewise from another circular point at infinity J one can draw tangents to the curve: $JQ_1, \ldots, JQ_m$. According to Plücker, any point F_{ij} of the intersection of IP_i and JQ_j is called a focus of the algebraic curve.

As a result, we have m^2 foci. Due to e.g. the coalescence of tangents, the number of foci can be reduced. Moreover, formally, if a curve is circular (one passing through I and J) its tangents at these points should enter into the Plücker construction. For some reason, they are not recognized as giving foci (Cayley's convention). A maximal number of finite and real foci is m. A bicircular quartic of class m has $m - 4$ finite and real foci while any Cartesian of class m has $m - 3$ finite and real foci.

3.6. Remarks on 'confocality'

While the term 'confocal curves' is quite comprehensible, the similar and generally accepted term 'confocal surfaces' is a misnomer. For example, nothing like a 'focus' exists for surfaces in CP^3. For non-degenerate quadrics, analogues of foci are conics called 'focal conics' or 'focales'.

3.7. Foci of conics

Consider an ellipse of semi-axes a and b ($a > b$). The standard coordinates of its foci are

$$[(a^2 - b^2)^{1/2}, 0], [-(a^2 - b^2)^{1/2}, 0], [0, \mathrm{i}(a^2 - b^2)^{1/2}], [0, -\mathrm{i}(a^2 - b^2)^{1/2}],$$

and similar formulae hold for the hyperbola. The parabola case is different. From $I(J)$, the circular point at infinity emanates two ($m = 2$) tangent lines to the parabola. One traverses C^2 and the other is the line at infinity. On applying the Plücker definition, we have the standard focus (Pappus) and three foci at infinity (two of them are I and J). The case of a circle is rather curious. It has a single four-fold focus provided we ignore Cayley's convention.

We note that all confocal central conics are inscribed in the same quadrilateral with vertices at common foci.

4. Inspirations

Darboux formally graduated from l'École Normale in the summer or fall of 1864 after three years of study. The first important date in his scientific biography is 1 August 1864. In those days, regular sessions of the Academy took place every Monday. During the session of 1 August 1864, Joseph Serret presented to the audience the abstract of the work *by Darboux actuellement élève de l'École Normale* . . . . The work was in fact his master's thesis and some excerpts entitled 'Remarks on the theory of orthogonal surfaces' were later published in *Comptes Rendus* [19].

No doubt, young Jean Gaston during his studies at l'École Normale selected orthogonal systems as his favorite scientific subject. But who inspired him directly (by discussions, suggestions, etc) with the enthusiasm for orthogonal systems of curves and surfaces?

I would simply answer: almost all prominent Paris mathematicians of that epoch. At least they were academicians: Chasles, Lamé, Bertrand, Serret and Bonnet. The point is that all of them had already published important papers on orthogonal systems of surfaces and probably all knew the young geometry enthusiast. Another crucial factor was that by this time, the most important discoveries in the theory and applications of orthogonal systems had been made by French geometers. The highlights before Darboux are as follows.

- Binet introduces in 1813 elliptic coordinates in E^3.
- Dupin publishes in 1813 his earlier (1807) beautiful and important result: curves of intersection of two coordinate surfaces of orthogonal system in E^3 are curvature lines for both surfaces.
- The transformation of the standard Laplace equation to general orthogonal coordinates is first effected by Lamé in 1834. This paper begins a long series of his publications devoted to the application of orthogonal coordinates in mathematical physics. In particular while discussing the equilibrium states of heat flow in solid bodies, he introduces the notion of (physical) isothermic surfaces (1837). In 1838, Lamé transforms the Laplace equation to elliptic coordinates and then breaks up (by variable separation) the transformed equation into three ordinary differential equations (Lamé equations). The summary of his remarkable studies is given in the book [20] published in 1859. Two years later, Darboux begins his studies.
- In 1844, Bertrand (later Darboux's first teacher), inspired by the earlier works of Lamé, discusses the so-called isothermic orthogonal systems in E^3, i.e. all orthogonal systems whose parametric surfaces are isothermic in a physical sense of the word. *Inter alia*, he proves the following theorem:

In an isothermic orthogonal system, a parametrization of any parametric surface by curvature lines is necessarily conformal.

This property is nothing other than the definition of an isothermic surface in the sense of classical differential geometry and modern soliton theory [21].

In about 1860, the most worked out example of orthogonal coordinates in E^3 was afforded by elliptic coordinates. In 1842, Carl Gustav Jacob Jacobi introduced general '*elliptische Coordinaten*' in E^n generalizing *mutatis mutandis* Binet's definition of elliptic coordinates, and this is why I use the term 'elliptic coordinates' for arbitrary n (not only for $n = 2$).

To construct novel interesting examples of orthogonal coordinates in E^3 was a true challenge for many geometers of those days, including the young, ambitious and gifted student Jean Gaston Darboux.

I believe that this section was of some help in answering question 1 above.

5. Reconstruction

Question 2 is a specific 'inverse problem'. I select three important papers by the young Darboux: his first paper [1], his master's thesis [19] and his doctor's thesis [6, 7]. The aim is to reconstruct Darboux's intellectual pathway from student status to researcher status mainly on the basis of these papers.

Prior to the discussion, I wish to 'decode' the comments by Eisenhart on [1] (see section 3).

The term 'double point' denotes either a node or a cusp. Any toric section is either bicircular quartic or a Cartesian with an axis of symmetry. Accordingly, its maximal class is $m = 8$; hence, the maximal total number of foci is 64. The Cayley convention reduces the number to 16 finite foci (among them, only $m - 4 = 4$ are finite and real). Darboux remarks that real foci are placed symmetrically w.r.t. the symmetry axis; thus, they are concyclic. Furthermore, 16 foci are situated on four circles and each circle contains four of them.

Any Cartesian is of at most $m = 6$ class; hence, the maximal number of foci is 36. The Cayley convention reduces the number to nine finite foci (among them, only $m - 3 = 3$ are finite and real).

I assume that from the very beginning, Darboux's goal is to construct novel, interesting, orthogonal coordinates in E^3 resembling elliptic coordinates in E^3. His (Chasles'?) first idea is to start with the simpler case of the plane. He is perfectly aware of the following important result of Kummer:

> A plane orthogonal system of curves consists of confocal (in the Plücker sense) curves.

It is natural to expect that, vice–versa, 'confocality' implies 'orthogonality'. Chasles—Darboux's mentor—remarked in [5] that Cartesian ovals have three collinear foci. Chasles did not give any proof of the statement and moreover he erroneously attributed a class $m = 8$ to Cartesian ovals. Darboux soon published an explicit construction of confocal Cartesian ovals which indeed form an orthogonal system of curves (see below).

A serious study of focal properties of toric sections and their relations to Cartesian ovals is the essence of [1]. Darboux believes that this study will help him to construct a confocal system of toric sections which will probably form an orthogonal system of curves. The final goal will be achieved when the resulting formulae is extended to three-space. This idea is almost exactly realized in [19, p 241]. He decides to work directly with three-space. The formulae of toric sections are replaced by their four-parameter central analogues in three-space:

$$(x^2 + y^2 + z^2)^2 + \alpha^2 x^2 + \beta^2 y^2 + \gamma^2 z^2 - h^4 = 0.$$

The requirement of orthogonality between two surfaces of this kind gives rise to an additional parameter labeling the parametric surfaces of the system, in perfect analogy with the elliptic case. The resulting orthogonal system is described in a compact form by a single formula (2) in [19, p 242]. If h goes to infinity, the novel orthogonal system approaches the elliptic one. The goal is achieved!

The parametric surfaces are the first examples of generalized (Darboux) cyclides. I call them 'central cyclides'. A general expression of Darboux cyclides reads as

$$u_0(x^2 + y^2 + z^2)^2 + u_1(x^2 + y^2 + z^2) + u_2 = 0.$$

Here u_i stands for a polynomial of ith order in x, y and z. The famous Dupin's cyclides are special cases. A proper treatment of cyclides is provided by the methods of algebraic geometry. In particular, when extended to CP^3 they can be characterized by their singularities at infinity.

In [6, p 61], Darboux comes back to the original line of reasoning. He writes down an equation of bicircular quartics in the form

$$(x^2 + y^2)^2 + u_1(x^2 + y^2) + u_2 = 0.$$

Then in the Note I to the main text of [6] he shows that they can form an orthogonal system of curves.

A remarkable connection between elliptic functions and plane orthogonal systems formed by cycliques of fourth order was established in 1860 by Siebeck [22]. For example, consider a map $(\xi, \eta) \to (x, y)$ given by $x + \mathrm{i}y = sn\,(\xi + \mathrm{i}\eta)$. One can directly compute images of lines $\xi = \text{const}$ and $\eta = \text{const}$. In both cases, they are central bicircular quartics and moreover they obviously form an orthogonal system. For excellent illustrations, see the book by Dixon [23].

It is not clear whether Darboux was aware of the results by Siebeck. In any case, he used Siebeck's methodology in 1867 to prove that Cartesian ovals can form an orthogonal system [24]. In the same paper, Darboux gave a proof of the Chasles theorem on three foci of Cartesian ovals. His proof is identical to that given earlier by Crofton [25]!

His doctorate is a complete presentation of both previous and completely new results including those not directly related to cyclidic coordinates. In particular, using the celebrated

Dupin theorem, Darboux proves that his central cyclides are isothermic. Today we know that all cyclides, and hence quadrics, are isothermic surfaces.

Summarizing, aged 22, at the end of his studies, Darboux found a four-parameter class of novel orthogonal systems in E^3. The analogy between Darboux coordinates and elliptic coordinates is indeed striking. Also note that any elliptic system when inverted in a sphere is a special rather degenerate Darboux system.

6. The strange day of 1 August 1864

The theory of bicircular quartics (Cartesians) and their two-dimensional analogues is one of the most fascinating chapters of old algebraic geometry. At one time, these were enthusiastically studied in France by Moutard, Mannheim, Laguerre, de la Gournerie and obviously by Darboux, while they received attention in Great Britain, thanks to the important works of Crofton, Casey, Cayley and others. Apart from Darboux, the most important contributions to the theory have been made by Théodore Florentin Moutard (1827–1901) and John Casey (1820–1891).

Probably Moutard began his studies on the subject around 1860. The guiding principle of his early studies was a problem: to find a curve C which is invariant w.r.t. some inversion in a circle K. The simplest case is: C—circle and K—arbitrary circle orthogonal to C. Following Moutard (1864), an invariant curve C is called 'anallagmatic' (from Greek: 'not changed'). Moutard seemed reluctant to publish and it is highly probable that he was aware that bicircular quartics are anallagmatic w.r.t. four mutually orthogonal circles. These are exactly the circles containing foci!

For both Darboux (22) and Moutard (37), the day 1 August 1864 was probably the most bizarre in their scientific biographies. On this day, during a weekly session of the Academy, two *consecutive* presentations were as follows: Serret presented the results of Darboux and then Bonnet presented the results of Moutard. All realized that it was the big news of the day. Both reports concerned the *same* subject: the outline of the theory of cyclides. Serret, feeling guilty about this unfortunate event, gave a speech the following Monday assuring an audience that the authors had worked on the subject independently. For Darboux, it was a very important statement: his report was his master's thesis!

My paper is only a modest review. Jean Gaston Darboux deserves something more: a true scientific biography which *inter alia* will describe his enormous impact on modern mathematical physics, in particular theory of integrable systems.

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Antoni Sym